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Turbulent pair dispersion and scalar diffusion

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A method of treating turbulent pair dispersion and scalar diffusion is presented. Use is made of Kraichnan's form of Richardson's diffusion equation by relating the turbulent pair diffusivity to single-time Eulerian velocity statistics (which are presumed known) by means of a statistical independence hypothesis. In this procedure the diffusivity itself is coupled to solutions of the diffusivity equation in a self-consistent way.

The method is applied to both two- and three-dimensional flow. In three-dimensional inertial-range and dissipative-range turbulence the turbulent pair diffusivity is determined and used to find the values of the coefficients of the scalar spectrum in the $k^{-\frac{5}{3}}$ and k^{-1} ranges with good agreement with experiment. The Obukhov–Corrsin constant is found to be 0·49 and the Batchelor constant is $\sqrt{5}$. In two-dimensional turbulence the results are compared with constant-pressure balloon dispersion experiments. Results are also found for the rate of decay of scalar intensity in the special case where the initial scalar spectrum peaks in the inertial range.

1. Introduction

The most important property of turbulent fluid motion is its ability to disperse fluid particles which were initially close together. This is of practical importance for the dispersal and dilution of pollutants in the environment and is also of fundamental importance to the nature of turbulence. Of equal practical importance, turbulence can mix different chemical species or fluids with non-uniform temperature in such a way that molecular processes such as diffusion or chemical reaction are enhanced.

In the processes of pair dispersion and scalar diffusion under consideration the statistics of the velocity field are presumed known and the statistics of marked fluid particles or concentrations of mixing species are sought. It has been known since G. I. Taylor's work (Taylor 1921; Corrsin 1962) that these problems are more natural if the Lagrangian statistics of the velocity field are known. However, the Eulerian statistics are more easily measured and are known in some idealized situations and it is therefore desirable to relate these quantities.

In the present paper turbulent dispersion and diffusion in an isotropic, homogeneous, incompressible turbulent flow field with zero mean velocity will be considered. Use will be made of Kraichnan's (1965, 1966*b*) form of Richardson's (1926) diffusion equation, which will be independently derived. Kraichnan's version of this equation was derived by means of the Lagrangian History Direct Interaction approximation and has a turbulent diffusivity term which is expressed in terms of a Lagrangian

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velocity correlation function. In the present work the Lagrangian and Eulerian statistics are related by means of an independence hypothesis, similar to that of Corrsin (1959, 1962), but which makes use of persistence of vorticity in order to obtain a result which depends only on the single-time Eulerian velocity correlation function. These approximations yield closed self-consistent equations which determine the turbulent pair diffusivity function. This is done in §§ 2 and 3.

In high-Reynolds-number turbulence, which is dominated by a broad inertial subrange, results very similar to those of Kraichnan (1966*b*) are obtained in § 4, except that the turbulent pair diffusivity in the present work is about half as large. This leads to a value of the Obukhov–Corrsin constant which is in better agreement with experiment.

The equations are also applied to the dissipative range with reasonable results and to the inertial range of two-dimensional turbulence where the results may be compared with balloon dispersal data.

In § 5 results on the pair distribution function are presented. Results on the scalar spectrum are found in § 6, and the rate of decay of scalar fluctuation intensity in § 7.

2. Derivation of the basic equations

Let $\mathbf{u}(\mathbf{x}, t)$ be one realization from an ensemble of incompressible velocity fields expressed in Eulerian co-ordinates. The trajectories $\mathbf{r}(t)$ of fluid particles in this field are provided by solutions of the ordinary differential equation

$$\left.\begin{aligned} \frac{d\mathbf{r}}{dt} &= \mathbf{u}(\mathbf{r}, t), \\ \mathbf{r} = \mathbf{X} \quad &\text{at} \quad t = s, \end{aligned}\right\} \tag{2.1}$$

which is the definition of 'velocity at a point'. Suppose that this is integrated both forward and backward in time and for all initial positions $\mathbf{X}$. Denote the solution by

$$\mathbf{r} = \mathbf{R}(\mathbf{X}, s|t),$$

a notation, due to Kraichnan (1965), which indicates explicitly the dependence on the initial conditions. The variable after the vertical bar is the current time, that before the bar is the labelling time. In words, $\mathbf{R}(\mathbf{X}, s|t)$ is the position at time t of the fluid particle which is at position $\mathbf{X}$ at time s. The function $\mathbf{R}$ has the property $\mathbf{R}(\mathbf{X}, s|s) \equiv \mathbf{X}$, and, when s and t are held fixed the relationship $\mathbf{x} = \mathbf{R}(\mathbf{X}, s|t)$ gives a transformation from the Lagrangian co-ordinates $\mathbf{X}$ to the Eulerian co-ordinates $\mathbf{x}$. It is assumed that this transformation is one to one and has a unique inverse given by $\mathbf{X} = \mathbf{R}(\mathbf{x}, t|s)$. The same notation also describes the inverse function because of the following tautology: $\mathbf{R}(\mathbf{x}, t|s)$ is the position at time s (the current time now) of the fluid particle which is at $\mathbf{x}$ at time t. But this position has been called $\mathbf{X}$.

A further property of the Lagrangian–Eulerian transformation is important. The Jacobian determinant, $J = \det(\partial x_i/\partial X_j)$, being the ratio of volume elements, is unity for incompressible flow.

In the formalism to be developed on the following pages considerable use will be made of Dirac's delta function. This is the generalized function defined, in the one-dimensional case, by the properties

$$\delta(x) = 0, \quad x \neq 0,$$

and

$$\int_{-\infty}^{\infty} \phi(x)\,\delta(x)\,dx = \phi(0),$$

for any continuous function $\phi(x)$. The latter characterization is called the sifting property. An important relation is

$$\delta[y-f(x)] = \frac{\delta[x-f^{-1}(y)]}{|df/dx|},$$

where it is assumed that $y = f(x)$ has a unique solution $x = f^{-1}(y)$ for each y.

A delta function with a three-dimensional vector argument is the product of three one-dimensional delta functions. The three-dimensional version of the above inversion relation is

$$\delta[\mathbf{y}-\mathbf{f}(\mathbf{x})] = \frac{\delta[\mathbf{x}-\mathbf{f}^{-1}(\mathbf{y})]}{\det(\partial f_i/\partial x_j)},$$

again assuming a unique solution to the equation $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for each $\mathbf{y}$. The particular application of this, which recurs throughout this work, is

$$\delta[\mathbf{x}-\mathbf{R}(\mathbf{X}, s|t)] = \delta[\mathbf{X}-\mathbf{R}(\mathbf{x}, t|s)], \tag{2.2}$$

which follows from the notation for the inverse of the Lagrangian–Eulerian transformation and the fact that the Jacobian of this transformation is unity.

The probability density for the positions $\mathbf{x}_1, \mathbf{x}_2$ of two fluid particles at time t, when their positions are known to be $\mathbf{X}_1$, $\mathbf{X}_2$ at time s, is defined by

$$P_2(\mathbf{x}_1, \mathbf{x}_2, t|\mathbf{X}_1, \mathbf{X}_2, s) = \langle \delta[\mathbf{x}_1 - R(\mathbf{X}_1, s|t)]\,\delta[\mathbf{x}_2 - \mathbf{R}(\mathbf{X}_2, s|t)]\rangle, \tag{2.3}$$

where the angle bracket indicates an ensemble average over all realizations of the velocity field. In this equation $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{X}_1$, $\mathbf{X}_2$ are all fixed numbers and the R functions are random variables through their dependence on the velocity field. When the arguments of the delta functions in (2.3) are inverted by using (2.2) it is seen that

$$P_2(\mathbf{x}_1, \mathbf{x}_2, t|\mathbf{X}_1, \mathbf{X}_2, s) \equiv P_2(\mathbf{X}_1, \mathbf{X}_2, s|\mathbf{x}_1, \mathbf{x}_2, t). \tag{2.4}$$

Therefore P_2 can also be regarded as the probability density for the positions $\mathbf{X}_1$, $\mathbf{X}_2$ of the two particles at time s when their positions $\mathbf{x}_1$, $\mathbf{x}_2$ are known at time t.

For homogeneous turbulence the probability density for the relative positions of two fluid particles is related to P_2 by the following definition and manipulations;

$$\begin{aligned}
&P_r(\mathbf{x}_2 - \mathbf{x}_1, t|\mathbf{X}_2 - \mathbf{X}_1, s)\\
&\quad\equiv \langle \delta[\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{R}(\mathbf{X}_2, s|t) + \mathbf{R}(\mathbf{X}_1, s|t)]\rangle\\
&\quad= \int \langle \delta[\mathbf{X}_1 - \boldsymbol{\Delta} - \mathbf{R}(\mathbf{X}_1, s|t)]\,\delta[\mathbf{x}_2 - \boldsymbol{\Delta} - \mathbf{R}(\mathbf{X}_2, s|t)]\rangle\, d\boldsymbol{\Delta}\\
&\quad= \int P_2(\mathbf{x}_1 - \boldsymbol{\Delta}, \mathbf{x}_2 - \boldsymbol{\Delta}, t|\mathbf{X}_1, \mathbf{X}_2, s)\, d\boldsymbol{\Delta}\\
&\quad= \int P_2(\mathbf{x}_1, \mathbf{x}_2, t|\mathbf{X}_1 + \boldsymbol{\Delta}, \mathbf{X}_2 + \boldsymbol{\Delta}, s)\, d\boldsymbol{\Delta}.
\end{aligned} \tag{2.5}$$

The third line follows from the sifting property of delta functions. The fifth line follows from the fourth by using homogeneity. Alternatively, this result may be obtained by changing variables in P_2 to obtain the joint probability density for the sum and difference of $\bar{\mathbf{x}}_2$ and $\bar{\mathbf{x}}_1$ and then find the marginal density P_r by integrating over the sum variable.

An equation for P_2 may be obtained by differentiating (2.3), which defines P_2, with respect to time and using (2.1). This gives

$$\frac{\partial P_2(\mathbf{x}_1,\mathbf{x}_2,t|\mathbf{X}_1,\mathbf{X}_2,s)}{\partial t} = -\sum_{j=1}^{2}\frac{\partial}{\partial \mathbf{x}_j}\cdot\left\langle \mathbf{u}(\mathbf{x}_j,t)\prod_{i=1}^{2}\delta[\mathbf{x}_i-\mathbf{R}(\mathbf{X}_i,s|t)]\right\rangle. \tag{2.6}$$

Kraichnan's diffusion equation may be obtained by using approximations similar to those used in mixing-length theories, where relationships of the form $\langle \mathbf{u}T\rangle = D\nabla\langle T\rangle$, for transportable quantities T, are obtained. The method described below makes use of some ideas from the theory of Markov processes in order to make the form of the diffusivity explicit. As a temporary notation let

$$\begin{aligned} f(\mathbf{x}_1,\mathbf{x}_2,t) &= \prod_{i=1}^{2}\delta[\mathbf{x}_i-\mathbf{R}(\mathbf{X}_i,s|t)] \\ &\equiv \prod_{i=1}^{2}\delta[\mathbf{X}_i-\mathbf{R}(\mathbf{x}_i,t|s)], \end{aligned} \tag{2.7}$$

where the second line follows from (2.2). This suppresses the dependence on $\mathbf{X}_1,\mathbf{X}_2,s$, which are held fixed. The quantity $f(\mathbf{x}_1,\mathbf{x}_2,t)$ is transportable, that is, its values are carried along by the natural motion of the fluid particles. This is shown by using the first form of (2.7) and (2.1) to calculate

$$\frac{\partial f}{\partial t}+\mathbf{u}(\mathbf{x}_1,t)\cdot\frac{\partial f}{\partial \mathbf{x}_1}+\mathbf{u}(\mathbf{x}_2,t)\cdot\frac{\partial f}{\partial \mathbf{x}_2}\equiv 0.$$

Because of this one may write

$$f(\mathbf{x}_1,\mathbf{x}_2,t) = f[\mathbf{R}(\mathbf{x}_1,t|t-\Delta t),\mathbf{R}(\mathbf{x}_2,t|t-\Delta t),t-\Delta t], \tag{2.8}$$

for arbitrary Δt. This expresses $f(\mathbf{x}_1,\mathbf{x}_2,t)$ in terms of its value at an earlier time, $t-\Delta t$, at the positions the fluid particles would have in order to be at $\mathbf{x}_1$ and $\mathbf{x}_2$ at time t. It is convenient now to write

$$\mathbf{R}(\mathbf{x}_i,t|t-\Delta t) = \mathbf{x}_i-\int_{t-\Delta t}^{t}\mathbf{u}(\mathbf{x}_i,t|T)\,dT,$$

where

$$\mathbf{u}(\mathbf{x}_i,t|T)\equiv\mathbf{u}[\mathbf{R}(\mathbf{x}_i,t|T),T]$$

is a shorter notation. This represents the change in position of a fluid particle as the integral of its velocity. Equation (2.8) may now be written

$$f(\mathbf{x}_1,\mathbf{x}_2,t) = \int f(\mathbf{x}_1-\boldsymbol{\Delta},\mathbf{x}_2-\boldsymbol{\Delta}_2,t-\Delta t)\prod_{i=1}^{2}\delta(\boldsymbol{\Delta}_i-\mathbf{S}_i)\,d\boldsymbol{\Delta}_1\,d\boldsymbol{\Delta}_2, \tag{2.9}$$

where

$$\mathbf{S}_i = \int_{t-\Delta t}^{t}\mathbf{u}(\mathbf{x}_i,t|T)\,dT.$$

Using equation (2.9) the angle bracket in equation (2.6) becomes

$$\langle\mathbf{u}(\mathbf{x}_j,t)f(\mathbf{x}_1,\mathbf{x}_2,t)\rangle = \int\left\langle f\mathbf{x}_1-\boldsymbol{\Delta}_1,\mathbf{x}_2-\boldsymbol{\Delta}_2,t-\Delta t)\,\mathbf{u}(\mathbf{x}_j,t)\prod_{i=1}^{2}\delta(\boldsymbol{\Delta}_i-\mathbf{S}_i)\right\rangle d\boldsymbol{\Delta}_1\,d\boldsymbol{\Delta}_2. \tag{2.10}$$

This is an exact expression and Δt is still an arbitrary time increment. From its definition the function $f(\mathbf{x}_1-\boldsymbol{\Delta}_1,\mathbf{x}_2-\boldsymbol{\Delta}_2,t-\Delta t)$ is a random variable which depends on the velocity field prior to the time $t-\Delta t$. It is now assumed that Δt is larger than the correlation time of the velocity field so that $f(t-\Delta t)$ will be statistically independent of the current time velocity, $\mathbf{u}(\mathbf{x}_j, t)$. The random variable S_i, defined in (2.9), depends on the velocity field in the time interval $t-\Delta t$ to t. Therefore the random variables $f(t-\Delta t)$ and S_i depend on the velocity field in non-overlapping time intervals. If the velocity field had vanishingly small correlation time, as for a Markoff process, they would be statistically independent. It will be assumed that they are approximately independent, this being partly justified because they depend only on time integrals of the velocity over their respective time intervals which has the effect of partially averaging the velocity, thus decreasing the fluctuations from trial to trial. Since the velocity $\mathbf{u}(\mathbf{x}_i, t)$ and S_i are more strongly correlated the specific assumption is that

$$\langle \mathbf{u}(\mathbf{x}_j, t) f(\mathbf{x}_1, \mathbf{x}_2, t)\rangle \simeq \int \langle f(\mathbf{x}_1-\mathbf{\Delta}_1, \mathbf{x}_2-\mathbf{\Delta}_2, t-\Delta t)\rangle \left\langle \mathbf{u}(\mathbf{x}_j, t) \prod_{i=1}^{2} \delta(\mathbf{\Delta}_i - \mathbf{S}_i)\right\rangle d\mathbf{\Delta}_1 \, d\mathbf{\Delta}_2, \tag{2.11}$$

where Δt is of the order of the velocity correlation time.

A second assumption is that the function $\langle f\rangle$ is a slowly varying function of its position and time arguments so that it may be expanded to first order as

$$\begin{aligned} &\langle f(\mathbf{x}_1-\mathbf{\Delta}_1, \mathbf{x}_2-\mathbf{\Delta}_2, t-\Delta t)\rangle \\ &\quad = \langle f(\mathbf{x}_1, \mathbf{x}_2, t)\rangle - \mathbf{\Delta}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} \langle f(\mathbf{x}_1, \mathbf{x}_2, t)\rangle \\ &\quad\quad - \mathbf{\Delta}_2 \cdot \frac{\partial}{\partial \mathbf{x}_2} \langle f(\mathbf{x}_1, \mathbf{x}_2, t)\rangle - \Delta t \frac{\partial}{\partial t} \langle f(\mathbf{x}_1, \mathbf{x}_2, t)\rangle + \ldots, \end{aligned}$$

this being strictly justified for large time. Substituting this into (2.11), carrying out the integration on $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ and using the fact that $\langle \mathbf{u}\rangle = 0$ and $\langle f\rangle = P_2$ gives the final result

$$\frac{\partial P_2}{\partial t} = \sum_{i,j=1}^{2} \frac{\partial}{\partial \mathbf{x}_j} \cdot \mathbf{D}_{ji} \cdot \frac{\partial}{\partial \mathbf{x}_i} P_2 \tag{2.12}$$

with

$$\mathbf{D}_{ji} = \int_s^t \langle \mathbf{u}(\mathbf{x}_j, t)\, \mathbf{u}(\mathbf{x}_i, t|T)\rangle \, dT. \tag{2.13}$$

In the last expression the lower limit of integration, which should be $t-\Delta t$, has been replaced by s. This is justified when $s < t-\Delta t$ since the velocities in the integrand are uncorrelated when their time arguments differ by more than Δt. (This is further discussed in the next paragraph for the case $t-s < \Delta t$. With this step, equation (2.12) becomes independent of Δt.

While the approximations made to derive (2.12) are better justified when $t-s$ is larger than the velocity correlation time, it will be used without this restriction. Some justification for believing that (2.12) is suitable for small $t-s$ can be found by using it to calculate the dispersion tensor. Multiplying (2.12) by $(\mathbf{x}_2-\mathbf{x}_1)(\mathbf{x}_2-\mathbf{x}_1)$ and integrating over $\mathbf{x}_1$ and $\mathbf{x}_2$ gives

$$\frac{\partial}{\partial t} \langle (\mathbf{x}_2-\mathbf{x}_1)(\mathbf{x}_2-\mathbf{x}_1)\rangle = \int (2\mathbf{D}_{22} + 2\mathbf{D}_{11} - 4\mathbf{D}_{12}) P_2 \, d\mathbf{x}_1 \, d\mathbf{x}_2.$$

Upon expanding the right-hand side of this for small $t-s$ one obtains

$$\langle(\mathbf{x}_2-\mathbf{x}_1)(\mathbf{x}_2-\mathbf{x}_1)\rangle$$
$$=(\mathbf{X}_2-\mathbf{X}_1)(\mathbf{X}_2-\mathbf{X}_1)+\langle(\mathbf{u}(\mathbf{X}_2,s)-\mathbf{u}(\mathbf{X}_1,s))(\mathbf{u}(\mathbf{X}_2,s)-\mathbf{u}(\mathbf{X}_1,s))\rangle(t-s)^2+\ldots,$$

which can be shown to be an exact result.

An equation for P_r may be obtained by noting, from the fifth line of equation (2.5), that P_r is obtained from P_2 by integrating over the initial positions; therefore P_r must also satisfy (2.12). Then using the fact that P_r depends on $\boldsymbol{\rho}=\mathbf{x}_2-\mathbf{x}_1$, and $\mathbf{D}_{ji}$ depends on $\mathbf{x}_j-\mathbf{x}_i$ by homogeneity, the equation for P_r may be written

$$\frac{\partial P_r}{\partial t}=\frac{\partial}{\partial\boldsymbol{\rho}}.2\mathbf{D}(\boldsymbol{\rho},t|s).\frac{\partial}{\partial\boldsymbol{\rho}}P_r \tag{2.14}$$

with

$$\mathbf{D}(\boldsymbol{\rho},t|s)=\int_s^t\langle\mathbf{u}(\mathbf{x}_1,t)[\mathbf{u}(\mathbf{x}_1,t|T)-\mathbf{u}(\mathbf{x}_2,t|T)]\rangle\,dT. \tag{2.15}$$

For homogeneous, isotropic, incompressible turbulence the turbulent pair diffusivity $\mathbf{D}$ is a symmetric tensor function of $\boldsymbol{\rho}=\mathbf{x}_2-\mathbf{x}_1$, with zero divergence on either index; $\partial D_{ij}/\partial\rho_i=\partial D_{ij}/\partial\rho_j=0$.

The expression for the turbulent pair diffusivity contains the Lagrangian two-point velocity correlation function $\langle\mathbf{u}(\mathbf{x}_1,t)\,\mathbf{u}(\mathbf{x}_2,t|T)\rangle$. This will be related to the single-time, two-point, Eulerian velocity correlation function $\langle\mathbf{u}(\mathbf{x}_1,t)\,\mathbf{u}(\mathbf{x}_2,t)\rangle$ which will be presumed given. The method to be employed is a variation of Corrsin's (1959, 1962) independence hypothesis. Corrsin's hypothesis has been used by Saffman (1962, Peskin (1974) and Lundgren & Pointin (1976) for single-particle dispersion, relating the single-point Lagrangian function to the two-point, two-time, Eulerian function as Corrsin originally proposed. Taylor & McNamara (1971), Montgomery (1975) and Salu & Montgomery (1977), in related plasma diffusion papers, made additional hypotheses which related the two-time Eulerian function to the single-time Eulerian function. The hypothesis made in the present work is related to these.

Some care must be taken in pair dispersion so that the turbulent pair diffusivity and hence P_r is invariant to random Galilean transformations (Kraichnan 1965). A random Galilean transformation is produced by adding to each realization of the velocity field a spatially and temporally constant independent random velocity with zero mean. Such a transformation obviously does not affect the relative motion of the fluid particles. An approximate theory which does not have this property will not properly treat the effect of the large eddies to simply convect two dispersing points when they are close together. For instance, the turbulent diffusivity should not depend on the probability density for the position of a single fluid particle since this is not invariant under the transformation.

The idea is to express the second quantity in the Lagrangian autocorrelation, $\langle\mathbf{u}(\mathbf{x}_1,t)\,\mathbf{u}(\mathbf{x}_2,t|T)\rangle$, in terms of the velocity field at the current time. For instance, if one could assume that the velocity of a fluid particle remains constant following its motion one would have $\mathbf{u}(\mathbf{x}_2,t|T)=\mathbf{u}(\mathbf{x}_2,t)$. This would make the Lagrangian autocorrelation equal the Eulerian single-time autocorrelation, a result which is only valid for small time differences. In the following a similar procedure will be pursued by using the transport properties of the vorticity field to relate $\mathbf{u}(\mathbf{x}_2 t|T)$ to the current velocity field.

The procedure begins by writing

$$\langle \mathbf{u}(\mathbf{x}_1, t)\, \mathbf{u}(\mathbf{x}_2, t|T)\rangle = \int \langle \mathbf{u}(\mathbf{x}_1, t)\, \mathbf{u}(\mathbf{X}_2, T)\, \delta[\mathbf{X}_2 - \mathbf{R}(\mathbf{x}_2, t|T)]\rangle\, d\mathbf{X}_2 \quad (2.16)$$

and then expressing $\mathbf{u}(\mathbf{X}_2, T)$ in terms of the vorticity, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, at the same time by

$$\mathbf{u}(\mathbf{X}_2, T) = \frac{1}{4\pi} \int \frac{\partial}{\partial \mathbf{X}_2} \frac{1}{|\mathbf{X}_2 - \mathbf{X}_3|} \times \boldsymbol{\omega}(\mathbf{X}_3, T)\, d\mathbf{X}_3. \quad (2.17)$$

The vorticity at time T is then expressed in terms of the vorticity at the current time t by means of Kelvin's theorem (for inviscid flow):

$$\boldsymbol{\omega}(\mathbf{X}_3, T) = \boldsymbol{\omega}(\mathbf{x}_3, t) . \frac{\partial \mathbf{R}(\mathbf{x}_3, t|T)}{\partial \mathbf{x}_3}, \quad (2.18)$$

where

$$\mathbf{x}_3 = \mathbf{R}(\mathbf{X}_3, T|t).$$

With these substitutions equation (2.16) becomes

$$\langle \mathbf{u}(\mathbf{x}_1, t)\, \mathbf{u}(\mathbf{x}_2, t|T)\rangle = -\frac{1}{4\pi} \int \mathbf{T} \times \frac{\partial}{\partial \mathbf{X}_2} \frac{1}{|\mathbf{X}_2 - \mathbf{X}_3|}\, d\mathbf{X}_2\, d\mathbf{X}_3\, d\mathbf{x}_3, \quad (2.19)$$

where

$$\mathbf{T} = \left\langle \mathbf{u}(\mathbf{x}_1, t)\, \boldsymbol{\omega}(\mathbf{x}_3, t) . \frac{\partial \mathbf{R}(\mathbf{x}_3, t|T)}{\partial \mathbf{x}_3} \delta[\mathbf{x}_2 - \mathbf{R}(\mathbf{X}_2, T|t)]\, \delta[\mathbf{x}_3 - \mathbf{R}(\mathbf{X}_3, T|t)] \right\rangle . \quad (2.20)$$

Thus things have been manipulated in such a way that the explicitly appearing velocity field depends on the current time t, while the functions $\mathbf{R}$ depend on the average velocity between times t and T. At this stage Corrsin's independence hypothesis is invoked with the assumption that the latter quantities are approximately independent of the former and reversed brackets are inserted at the position of the dot in the scalar product. This would be exact if the velocity field were a Markov process, and is expected to be a very good approximation when $t-T$ is much larger than a velocity correlation time. On the other hand it is exact for small $t-T$ for then $\mathbf{R}(\mathbf{X}, T|t)$ becomes exactly independent of the velocity fields, so the approximation could be more generally valid.

In addition to the above approximation it will be assumed that the vortex stretching function $\partial\mathbf{R}/\partial\mathbf{x}_3$ is statistically independent of the $\mathbf{R}$ functions which appear in the delta functions, an approximation which is true for small $t-T$ because $\mathbf{R} \rightarrow \mathbf{x}_3$ in this limit. For larger values of $t-T$ one can estimate the dependence of $\mathbf{R}$ upon $\mathbf{x}_3$ by the following reasoning. $\mathbf{R}$ depends on the average velocity along a particle trajectory. Consider two neighbouring trajectories. The difference in $\mathbf{R}$ depends on the average of the velocity difference seen by these two particles and therefore depends only on the smallest eddies in the flow. Therefore $\partial\mathbf{R}/\partial\mathbf{x}_3$ is a statistical quantity which fluctuates strongly with the scales of the smallest eddies of the velocity field, while $\mathbf{R}$ itself depends most strongly on the largest eddies. These quantities are therefore approximately independent. For two-dimensional flows, of course, the stretching function does not occur.

With these independence assumptions, the definition of P_2 and $\langle \partial\mathbf{R}/\partial\mathbf{x}_3\rangle = \mathbf{I}$, the expression for $\mathbf{T}$ becomes

$$\mathbf{T} = \langle \mathbf{u}(\mathbf{x}_1, t)\, \boldsymbol{\omega}(\mathbf{x}_3, t)\rangle\, P_2(\mathbf{x}_2, \mathbf{x}_3, t|\mathbf{X}_2, \mathbf{X}_3, T). \quad (2.21)$$

When this is substituted into (2.19), a change of variable to $\mathbf{q} = \mathbf{X}_3 - \mathbf{X}_2$ made, with $\mathbf{X}_2$ fixed, and the last line of (2.5) used, followed by integration by parts and the use of isotropy, one obtains

$$\langle \mathbf{u}(\mathbf{x}_1, t)\,\mathbf{u}(\mathbf{x}_1 + \boldsymbol{\rho}, t|T)\rangle = \int \mathbf{U}(\boldsymbol{\xi} + \boldsymbol{\rho}, t)\, Q(\boldsymbol{\xi}, t|T)\, d\boldsymbol{\xi}, \tag{2.22}$$

where

$$\mathbf{U}(\boldsymbol{\rho}, t) = \langle \mathbf{u}(\mathbf{x}_1, t)\,\mathbf{u}(\mathbf{x}_1 + \boldsymbol{\rho}, t)\rangle, \tag{2.23}$$

and

$$Q(\boldsymbol{\xi}, t|T) = -\frac{1}{4\pi}\frac{\partial}{\partial \boldsymbol{\xi}} \cdot \int \left(\frac{\partial}{\partial \mathbf{q}}\frac{1}{q}\right) P_r(\boldsymbol{\xi}, t|\mathbf{q}, T)\, d\mathbf{q}. \tag{2.24}$$

This is the desired result which expresses the Lagrangian velocity correlation function in terms of the Eulerian two-point single-time correlation function. This is not simply a linear relation between the two functions, because Q depends on P_r which is a solution of (2.14). The relationship may be seen more clearly by substituting (2.22) into (2.15) for the turbulent diffusivity. This gives

$$\mathbf{D}(\boldsymbol{\rho}, t|s) = \int [\mathbf{U}(\boldsymbol{\xi}, t) - \mathbf{U}(\boldsymbol{\xi} + \boldsymbol{\rho}, t)] \int_s^t Q(\boldsymbol{\xi}, t|T)\, dT\, d\boldsymbol{\xi}. \tag{2.25}$$

Therefore one sees that, while $\mathbf{D}$ depends explicitly on the Eulerian function at the current time, P_r and Q will depend on the entire past history of this function (stationary turbulence has not yet been assumed). Also it is clear that the condition of invariance under a random Galilean transformation is satisfied, since the only place where the velocity field appears is in $\mathbf{U}$ and the difference form in (2.25) is invariant ($\mathbf{U}$ itself is not invariant).

Kelvin's theorem, which is only true for inviscid flow, was used in the above argument. One might question the neglect of viscosity. The process of pair dispersion depends on eddies of the same size as the spacing between the points. If the spacing is large compared with the Kolmogorov microscale the viscosity-dominated small eddies should have no effect and can be neglected. When the spacing is small, in the dissipation range, say, it is believed that (2.25) is still substantially correct (with an appropriate viscosity-dominated form for $\mathbf{U}$), because, while vorticity will tend to diffuse, the integrated vorticity of a blob will be conserved so that there will tend to be a certain amount of persistence of vorticity. It will be seen below that this assumption leads to quite reasonable results.

The above equations were derived for three-dimensional flow. Some small modifications are required for two-dimensional flow; a logarithmic potential in (2.17) and no stretching in (2.18). In the final results Q is modified by replacing $-(4\pi q)^{-1}$ in (2.24) by $\ln q/2\pi$ and all integrals are to be taken as two-dimensional.

3. Equations for the turbulent pair diffusivity

The basic equations for the turbulent diffusivity are (2.25) with Q given by (2.24) and P_r by (2.14). It is possible to recast these as equations for Q. Two formulations are given, one in physical space, and one in wavenumber space, which is more convenient for some purposes.

3.1. *Physical-space formulation*

For isotropic homogeneous turbulence the turbulent diffusivity $\mathbf{D}$ and the Eulerian velocity correlation function $\mathbf{U}$, each having zero divergence, may be represented in terms of single-scalar 'longitudinal' functions,

$$\mathbf{D}(\boldsymbol{\rho}, t|s) = f(\rho, t|s)\frac{\boldsymbol{\rho}\boldsymbol{\rho}}{\rho^2} + g(\rho, t|s)\left(\mathbf{I} - \frac{\boldsymbol{\rho}\boldsymbol{\rho}}{\rho^2}\right), \tag{3.1}$$

$$g = f + \tfrac{1}{2}\rho\frac{\partial f}{\partial \rho}, \tag{3.2}$$

$$\mathbf{U}(\boldsymbol{\rho}, t) = h(\rho, t)\frac{\boldsymbol{\rho}\boldsymbol{\rho}}{\rho^2} + l(\rho, t)\left(\mathbf{I} - \frac{\boldsymbol{\rho}\boldsymbol{\rho}}{\rho^2}\right), \tag{3.3}$$

$$l = h + \tfrac{1}{2}\rho\frac{\partial h}{\partial \rho}, \tag{3.4}$$

where $h(\rho, t)$ can be regarded as known.

Define a function $\mathbf{W}(\boldsymbol{\xi}, t|T)$ by

$$\mathbf{W}(\boldsymbol{\xi}, t|T) = -\frac{1}{4\pi}\int\left(\frac{\partial}{\partial \mathbf{q}}\frac{1}{q}\right) P_r(\boldsymbol{\xi}, t|\mathbf{q}, T)\,d\mathbf{q};$$

then by (2.24)

$$Q(\xi, t|T) = \frac{\partial}{\partial \boldsymbol{\xi}}.\mathbf{W}(\boldsymbol{\xi}, t|T).$$

By isotropy,

$$\mathbf{W}(\boldsymbol{\xi}, t|T| = \hat{\boldsymbol{\xi}}\,W(\xi, t|T) \tag{3.7}$$

where $\hat{\boldsymbol{\xi}}$ is a unit vector and both Q and W depend only on the magnitude of $\boldsymbol{\xi}$. It is observed that the function $\mathbf{W}$ satisfies the same differential equation as P_r, namely

$$\frac{\partial}{\partial t}\hat{\boldsymbol{\rho}}\,W(\rho, t|s) = 2\mathbf{D}\colon\frac{\partial}{\partial \boldsymbol{\rho}}\frac{\partial}{\partial \boldsymbol{\rho}}\hat{\boldsymbol{\rho}}\,W(\rho, t|s). \tag{3.8}$$

Upon taking the scalar product with $\hat{\boldsymbol{\rho}}$ and using (3.1), this may be written

$$\frac{\partial W}{\partial t} = 2f\frac{\partial^2 W}{\partial \rho^2} + 2\left(2f + \rho\frac{\partial f}{\partial \rho}\right)\frac{1}{\rho}\left(\frac{\partial W}{\partial \rho} - \frac{W}{\rho}\right), \tag{3.9}$$

where Q is to be determined from

$$Q(\rho, t|s) = \frac{\partial}{\partial \boldsymbol{\rho}}.\hat{\boldsymbol{\rho}}\,W(\rho, t|s)$$
$$= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\rho^2 W. \tag{3.10}$$

The initial conditions on Q and W may be obtained from (2.24) and (3.5) by noting $P_r(\boldsymbol{\xi}, t|\mathbf{q}, T) \to \delta(\boldsymbol{\xi} - \mathbf{q})$ as $t \to T$; therefore

$$Q(\rho, t|s) \to \delta(\boldsymbol{\rho}) \quad \text{as} \quad t \to s \tag{3.11}$$

and

$$W(\rho, t|s) \to \frac{1}{4\pi\rho^2} \quad \text{as} \quad t \to s. \tag{3.12}$$

Equation (3.9) may be rewritten as a differential equation for the quantity $4\pi\rho^2 W$, namely

$$\left.\begin{aligned}\frac{\partial}{\partial t}4\pi\rho^2 W &= 2\left\{\frac{\partial}{\partial\rho}f\frac{\partial}{\partial\rho}-2f\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{3}{\rho}\frac{\partial f}{\partial\rho}\right\}4\pi\rho^2 W,\\ 4\pi\rho^2 W &\to 1 \quad \text{as} \quad t\to s.\end{aligned}\right\} \tag{3.14}$$

In this equation the function f is a functional of Q or W which can be obtained from (2.25) and (3.1). One way is by using

$$3f+\rho\frac{\partial f}{\partial\rho} \equiv D_{ii}(\boldsymbol{\rho},t) = \int[U_{ii}(\boldsymbol{\xi},t)-U_{ii}(\boldsymbol{\xi}+\boldsymbol{\rho},t)]\int_s^t Q(\xi,t|T)\,dT\,d\boldsymbol{\xi}. \tag{3.15}$$

Thus equations (3.14) and (3.15) are coupled equations for the two scalar quantities Q and f.

For two-dimensional flow there are some differences in the final equations. The vector relations (3.1) and (3.3) are correct but the latitudinal functions g and l are related to the longitudinal functions by

$$g = f+\rho\frac{\partial f}{\partial\rho}, \quad l = h+\rho\frac{\partial h}{\partial\rho}. \tag{3.16}$$

The function $\mathbf{W}$ is defined by

$$\mathbf{W}(\boldsymbol{\xi},t|T) = \frac{1}{2\pi}\int\left(\frac{\partial}{\partial\mathbf{q}}\ln q\right)P_r(\boldsymbol{\xi},t|\mathbf{q},T)\,d\mathbf{q}. \tag{3.17}$$

The final equations are

$$\left.\begin{aligned}\frac{\partial}{\partial t}2\pi\rho W &= 2\left\{\frac{\partial}{\partial\rho}f\frac{\partial}{\partial\rho}-f\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{2}{\rho}\frac{\partial f}{\partial\rho}\right\}2\pi\rho W,\\ 2\pi\rho W &\to 1 \quad \text{as} \quad t\to s,\end{aligned}\right\} \tag{3.18}$$

$$2f+\rho\frac{\partial f}{\partial\rho} \equiv D_{ii}(\rho,t) = \int[U_{ii}(\boldsymbol{\xi},t)-U_{ii}(\boldsymbol{\xi}+\boldsymbol{\rho},t)]\int_s^t Q(\xi,t|T)\,dT\,d\boldsymbol{\xi}, \tag{3.19}$$

where Q and W are related by

$$\begin{aligned}Q(\xi,t|T) &= \frac{\partial}{\partial\boldsymbol{\xi}}.\hat{\boldsymbol{\xi}}\,W(\xi,t|T)\\ &= \frac{1}{\xi}\frac{\partial}{\partial\xi}\xi W.\end{aligned} \tag{3.20}$$

3.2. *Spectral formulation*

The energy spectrum tensor is defined by

$$\boldsymbol{\Phi}(\mathbf{k},t) = \int e^{-i\mathbf{k}\cdot\boldsymbol{\rho}}\,\mathbf{U}(\boldsymbol{\rho},t)\,d\boldsymbol{\rho}. \tag{3.21}$$

Because of incompressibility and isotropy this may be represented in terms of a single scalar by

$$\boldsymbol{\Phi} = \tfrac{1}{2}\Phi_{\alpha\alpha}\left(\mathbf{I}-\frac{\mathbf{kk}}{k^2}\right), \tag{3.22}$$

where $\Phi_{\alpha\alpha}$ is related to the energy spectrum function E by

$$\left.\begin{aligned}\Phi_{\alpha\alpha}(k,t) &= 4\pi^2 E(k,t)/k^2,\\ \tfrac{1}{2}\langle\mathbf{u}(\mathbf{x}_1,t).\mathbf{u}(\mathbf{x}_1,t)\rangle = \tfrac{1}{2}U_{\alpha\alpha}(0,t) &= \int_0^\infty E(k,t)\,dk.\end{aligned}\right\} \tag{3.23}$$

Define

$$R(k,t|T) = \int e^{-i\mathbf{k}.\boldsymbol{\rho}} Q(\rho,t|T)\,d\boldsymbol{\rho}; \tag{3.25}$$

then from (2.22) one obtains

$$\int e^{-i\mathbf{k}.\boldsymbol{\rho}} \langle \mathbf{u}(\mathbf{x}_1,t)\,\mathbf{u}(\mathbf{x}_1+\boldsymbol{\rho},t|T)\rangle\,d\boldsymbol{\rho} = \boldsymbol{\Phi}(\mathbf{k},t)\,R(k,t|T). \tag{3.26}$$

Now denote

$$\mathbf{N}(\mathbf{k},t|s) = \frac{1}{(2\pi)^3}\int e^{-i\mathbf{k}.\boldsymbol{\rho}} \int_s^t \langle \mathbf{u}(\mathbf{x}_1,t)\,\mathbf{u}(\mathbf{x}_1+\boldsymbol{\rho},t|T)\rangle\,\lambda T\,d\boldsymbol{\rho}. \tag{3.27}$$

Clearly,

$$\mathbf{N}(\mathbf{k},t|s) = \tfrac{1}{2}N_{\alpha\alpha}\left(\mathbf{I}-\frac{\mathbf{kk}}{k^2}\right) \tag{3.28}$$

with

$$N_{\alpha\alpha}(k,t|s) = \frac{1}{(2\pi)^3}\Phi_{\alpha\alpha}(k,t)\int_s^t R(k,t|T)\,dT. \tag{3.29}$$

As a temporary notation introduce

$$P'_r(\mathbf{k},t|\mathbf{q},s) = \int e^{-i\mathbf{k}.\boldsymbol{\rho}} P_r(\boldsymbol{\rho},t|\mathbf{q},s)\,d\boldsymbol{\rho} \tag{3.30}$$

and transform (2.14). Using equation (3.27) this gives

$$\frac{\partial P'_r(\mathbf{k},t|\mathbf{q},s)}{\partial t} = 2\mathbf{kk}\mathbin{:}\int \mathbf{N}(\mathbf{p},t|s)\,[P'_r(\mathbf{k}-\mathbf{p},t|\mathbf{q},s)-P'_r(\mathbf{k},t|\mathbf{q},s)\,d\mathbf{p}. \tag{3.31}$$

Now transform (2.24), using (3.25). This yields

$$R(k,t|s) = \mathbf{k}.\left\{-\frac{i}{4\pi}\int\left(\frac{\partial}{\partial\mathbf{q}}\frac{1}{q}\right)P'_r(\mathbf{k},t|\mathbf{q},s)\,d\mathbf{q}\right\}. \tag{3.32}$$

The vector quantity in parentheses satisfies the same equation as P'_r, namely (3.31). Further, by istropy, it is equal to $\mathbf{k}R(k,t|s)/k^2$. By making this substitution, taking the scalar product with $\mathbf{k}$ and using (3.28) and (3.29) one obtains

$$\left.\begin{aligned}\frac{\partial R(k,t|s)}{\partial t} &= \frac{k^2}{2\pi}\int\frac{E(p,t)}{p^2}\int_s^t R(p,t|T)\,dT\left(1-\frac{(\mathbf{k}.\mathbf{p})^2}{k^2p^2}\right)B\,d\mathbf{p},\\ B &= \frac{\mathbf{k}.(\mathbf{k}-\mathbf{p})}{|\mathbf{k}-\mathbf{p}|^2}R(|\mathbf{k}-\mathbf{p}|,t|s)-R(k,t|s),\end{aligned}\right\} \tag{3.33}$$

a single nonlinear differential-integral equation for R, to be solved with the initial condition

$$R(k,s|s) = 1.$$

The longitudinal turbulent diffusivity f may be determined from R by using $f = \boldsymbol{\rho\rho}\mathbin{:}\mathbf{D}/\rho^2$ from equation (3.1), the definition of $\mathbf{D}$ from (2.25) and the inverse transforms of (3.25) and (3.27). The result is

$$f(\rho,t|s) = 2\int_s^t dT\int_0^\infty E(k,t)\,R(k,t|T)\left(\frac{1}{3}-\frac{\sin k\rho}{(k\rho)^3}+\frac{\cos k\rho}{(k\rho)^2}\right)dk. \tag{3.34}$$

The integration in (3.33) may be expressed in more convenient form by a standard change of variables (see Orszag 1974; Rose & Sulem 1978). It is noted that the integrand depends only on p and the angle θ between $\mathbf{p}$ and $\mathbf{k}$. In place of the angle variable the new variable $q = |\mathbf{k}-\mathbf{p}|$ is introduced. Using $d\mathbf{p} = 2\pi(qp/k)\,dq\,dp$, equation (3.33) can be written

$$\frac{\partial R(k,t|s)}{\partial t} = \int_{\Delta_k} \frac{E(p,t)}{p^2} \int_s^t R(p,t|T)\,dT \sin^2\theta Bkpq\,dp\,dq, \tag{3.35}$$

where

$$B = \frac{1}{2}\frac{k^2+q^2-p^2}{q^2}R(q,t|s) - R(k,t|s),$$

$$\sin^2\theta = 1 - \frac{(\mathbf{k}.\mathbf{p})^2}{k^2p^2}$$

$$= 1 - \left(\frac{q^2-k^2-p^2}{2kp}\right)^2,$$

and the region of integration Δ_k is

$$0 \leqslant p \leqslant \infty, \quad |p-k| \leqslant q \leqslant p+k.$$

For two-dimensional flow, equation (3.35) should be modified by replacing $\sin^2\theta$ by $4\sin\theta/\pi$ and (3.34) should be replaced by

$$f(\rho,t|s) = \int_s^t dT \int_0^\infty E(k,t)\,R(k,t|T)\left(1 - \frac{2J_1(k\rho)}{k\rho}\right)dk. \tag{3.36}$$

4. Solutions for the turbulent pair diffusivity

The formulations of the last section are for generally decaying homogeneous, isotropic turbulence when the energy-spectrum function is known. There are some special cases where the turbulence is stationary and the spectrum or the velocity correlation function have well-established simple forms where similarity solutions can be found and the turbulent pair diffusivity thus determined. These are worked out in the following subsections.

4.1. *The dissipation range*

For small ρ a well-known expansion of the Eulerian velocity correlation function is (Monin & Yaglom 1975, equation (21.16))

$$h = U^2 - \frac{\epsilon\rho^2}{30\nu} + \dots$$

and

$$\mathsf{U} = U^2\mathsf{I} + \frac{\epsilon}{30\nu}(\boldsymbol{\rho}\boldsymbol{\rho} - 2\rho^2\mathsf{I}), \tag{4.1}$$

where $3U^2 = \langle \mathbf{u}(\mathbf{x}_1,t).\mathbf{u}(\mathbf{x}_1,t)\rangle$ and ϵ is the energy dissipation rate. For high-Reynolds-number turbulence this requires that ρ be much smaller than the Kolmogorov microscale $(\nu^3/\epsilon)^{\frac{1}{4}}$, however for lower Reynolds number it should be valid under less stringent conditions, say ρ smaller than the integral scale. For stationary turbulence ϵ is constant.

It will be assumed that (3.14) and (3.15) have a solution which is dominated by this part of the Eulerian function for small ρ. If this is the case the solution can only depend on ρ, $t-s$ and the quantity $(\nu/\epsilon)^{\frac{1}{2}}$ which has units of time (it is the turnaround time of eddies of the size of the Kolmogorov microscale). Dimensional considerations therefore imply

$$f = (\epsilon/\nu)^{\frac{1}{2}}\rho^2 f_0(\tau), \quad 4\pi\rho^2 W = W_0(\tau), \tag{4.2}$$

where $\tau = (\epsilon/\nu)^{\frac{1}{2}}(t-s)$ and $W_0(0) = 1$. Equation (3.14) gives

$$\frac{dW_0(\tau)}{d\tau} = -12 f_0(\tau) W_0(\tau), \tag{4.3}$$

while (3.10) gives the simple result

$$Q = \frac{\partial}{\partial \boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}} \frac{W_0(\tau)}{4\pi\rho^2} = W_0(\tau)\,\delta(\boldsymbol{\rho}). \tag{4.4}$$

When this is substituted into (3.15) one obtains

$$f_0(\tau) = \frac{1}{30}\int_0^\tau W_0(\tau')\,d\tau'. \tag{4.5}$$

Equations (4.3) and (4.5) are easily solved for

$$f_0(\tau) = \frac{1}{6\sqrt{5}} \tanh(\tau/\sqrt{5}), \tag{4.6}$$

$$W_0(\tau) = 1/\cosh^2(\tau/\sqrt{5}). \tag{4.7}$$

The result

$$f = (\epsilon/\nu)^{\frac{1}{2}} \rho^2 f_0(\tau)$$

or more generally

$$\mathbf{D} = (\epsilon/\nu)^{\frac{1}{2}} \rho^2 f_0(\tau) \left(2\mathbf{I} - \frac{\boldsymbol{\rho}\boldsymbol{\rho}}{\rho^2}\right) \tag{4.8}$$

shows that the turbulent pair diffusivity becomes independent of time after several Kolmogorov times since $f_0 \rightarrow (6\sqrt{5})^{-1}$.

The Lagrangian correlation function can be obtained from (2.22) in the form

$$\langle \mathbf{u}(\mathbf{x}_1, t)\,\mathbf{u}(\mathbf{x}_1 + \boldsymbol{\rho}, t|s)\rangle - \langle \mathbf{u}(\mathbf{x}_1, t)\,\mathbf{u}(\mathbf{x}_1, t|s)\rangle = [\mathbf{U}(\boldsymbol{\rho}) - \mathbf{U}(0)]\,W_0(\tau). \tag{4.9}$$

In words this says that the correlation between the velocities at a fixed point and at the earlier position of a neighbouring point minus the correlation between the velocities at the fixed point and at its own earlier position equals the difference between the corresponding unlagged correlation functions times a decaying function of the lag time. The function W_0 decays to zero in several Kolmogorov times. Notice that these results have not been applied directly to (2.22) with $\boldsymbol{\rho} = 0$. That would be incorrect since $\langle \mathbf{u}(\mathbf{x}_1, t)\,\mathbf{u}(\mathbf{x}_1, t|s)\rangle$ depends on the energy-containing scales of the motion, not the dissipative scales.

4.2. *The two-dimensional inertial range*

For two-dimensional turbulence, which to a certain extent models the large-scale motions of the atmosphere, Kraichnan (1971) has proposed that the energy spectrum function behaves like $E \sim k^{-3}(\ln k/k_1)^{-\frac{1}{3}}$ for large k, decaying slightly faster than k^{-3} and Saffman (1971) proposes $E \sim k^{-4}$. Under such circumstances it is easy to see that the longitudinal correlation function behaves like

$$h = h_0 - a^{-2}\rho^2 + \ldots \tag{4.10}$$

and therefore

$$\mathbf{U} = h_0\mathbf{I} + 2a^{-2}\rho^2\left(\frac{\boldsymbol{\rho}\boldsymbol{\rho}}{\rho^2} - \tfrac{3}{2}\mathbf{I}\right) + \ldots, \tag{4.11}$$

where a is a constant with units of time. (If $E \sim k^{-3}$ an additional $\rho^2 \ln \rho$ term is needed in (4.10).)

As in the last subsection there will be a similarity solution of form

$$f = a^{-1}\rho^2 f_1(\tau_1), \quad 2\pi\rho W = W_1(\tau_1), \tag{4.12}$$

where $\tau_1 = (t-s)/a$ and $W_1(0) = 1$. Equation (3.18) then gives

$$\frac{dW_1(\tau_1)}{d\tau_1} = -8f_1(\tau_1)\, W_1(\tau_1), \tag{4.13}$$

and (3.20) and (3.19) give

$$Q = W_1(\tau_1)\,\delta(\boldsymbol{\rho}), \tag{4.14}$$

and

$$f_1(\tau_1) = \int_0^{\tau_1} W_1(\tau')\, d\tau'. \tag{4.15}$$

The solution of (4.13) and (4.15) is

$$f_1(\tau_1) = \tfrac{1}{2}\tanh(2\tau_1), \tag{4.16}$$

$$W_1(\tau_1) = 1/\cosh^2(2\tau_1), \tag{4.17}$$

and the final result is

$$f = a^{-1}\rho^2 f_1(\tau_1) \tag{4.18}$$

or

$$\mathbf{D} = a^{-1}\rho^2 f_1(\tau_1)\left(3\mathbf{I} - 2\frac{\boldsymbol{\rho\rho}}{\rho^2}\right). \tag{4.19}$$

The equation for the Lagrangian correlation function is the same as (4.9).

4.3. *The three-dimensional inertial range*

For large-Reynolds-number turbulence the Kolmogorov (1941) cascade theory gives a longitudinal velocity correlation function which behaves like

$$h = U^2 - b\epsilon^{\frac{2}{3}}\rho^{\frac{2}{3}} + \ldots \tag{4.20}$$

for ρ small, but large compared to the Kolmogorov microscale, $(\nu^3/\epsilon)^{\frac{1}{4}}$, which shrinks to zero when the Reynolds number goes to infinity. The energy spectrum function, E, related to h by

$$\Phi_{\alpha\alpha} = 4\pi^2 E/k^2 = 4\pi \int_0^\infty \frac{\sin k\rho}{k\rho}\frac{\partial}{\partial\rho}\rho^3 h\, d\rho, \tag{4.21}$$

is

$$E = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{4.22}$$

for large k, where C is the Kolmogorov constant. The constants C and b are related by $b = 0{\cdot}657C$.

In the physical-space formulation it is assumed that there is a solution which only depends on $\rho, t-s$ and ϵ. On dimensional grounds the dependent variables can only occur in the combination

$$y = \frac{\rho}{\tilde{\epsilon}^{\frac{1}{2}}(t-s)^{\frac{3}{2}}}, \tag{4.23}$$

where $\tilde{\epsilon} = C^{\frac{3}{2}}\epsilon$, and $4\pi\rho^2 W$ which is dimensionless and f which has units 'length²/time' must have the form

$$4\pi\rho^2 W = \tilde{W}(y), \tag{4.24}$$

$$f = \tilde{\epsilon}^{\frac{1}{3}}\rho^{\frac{4}{3}}\tilde{f}(y). \tag{4.25}$$

The solution in this formulation has not been completed. However, by means of (3.14) and (3.15) it can be established that

$$\tilde{W}(y) = \tilde{W}_0 y^3 + \tilde{W}_1 y^{\frac{11}{3}} + \dots, \tag{4.26}$$

$$\tilde{f}(y) = \tilde{f}_0 + \tilde{f}_1 y^{\frac{2}{3}} + \dots \tag{4.27}$$

for small y where the coefficients are constants. From equation (3.10)

$$Q = \frac{1}{4\pi\rho^3} y \frac{d\tilde{W}(y)}{dy} = \frac{1}{4\pi\rho^3}(3\tilde{W}_0 y^3 + \tfrac{11}{3}\tilde{W}_1 y^{\frac{11}{3}} + \dots). \tag{4.28}$$

In the spectral formulation given by (3.35) the function $R(k, t|s)$ can only depend on k, $t-s$ and ϵ; therefore it must be of the form

$$R(k, t|s) = R(x), \tag{4.29}$$

$$x = \tilde{\epsilon}^{\frac{1}{2}} k (t-s)^{\frac{3}{2}}. \tag{4.30}$$

By using (3.25) and the result for Q given by (4.28) it can be shown that

$$R(x) \sim x^{-\frac{11}{3}} \tag{4.31}$$

for large x.

By using (4.29) and (4.30), equation (3.35) takes the form

$$\frac{dR(x)}{dx} = \frac{4}{9}\int_{\Delta x} u^{-\frac{10}{3}} v \int_0^u \zeta^{-\frac{1}{3}} R(\zeta)\, d\zeta \sin^2\theta B\, du\, dv, \tag{4.32}$$

where $u = \tilde{\epsilon}^{\frac{1}{2}} p(t-s)^{\frac{3}{2}}$ and $v = \tilde{\epsilon}^{\frac{1}{2}} q(t-s)^{\frac{3}{2}}$ and

$$B = \frac{1}{2}\frac{x^2 + v^2 - u^2}{q^2} R(v) - R(x),$$

$$\sin^2\theta = 1 - \left(\frac{v^2 - x^2 - u^2}{2xu}\right)^2.$$

This is to be solved with the initial condition $R(0) = 1$.

The longitudinal diffusivity function $\tilde{f}$ defined by (4.25), may be expressed in terms of $R(x)$ by means of (3.34) which takes the convergent form

$$\tilde{f}(y) = \tfrac{4}{3} y^{-\frac{4}{3}} \int_0^\infty x^{-\frac{7}{3}} \int_0^x \zeta^{-\frac{1}{3}} R(\zeta)\, d\zeta \left(\frac{1}{3} - \frac{\sin xy}{(xy)^3} + \frac{\cos xy}{(xy)^2}\right) dx. \tag{4.33}$$

The related latitudinal function defined by $g = \tilde{\epsilon}^{\frac{1}{3}} \rho^{\frac{4}{3}} \tilde{g}(y)$ is given by

$$\tilde{g}(y) = \tfrac{5}{3}\tilde{f}(y) + \tfrac{1}{2} y \frac{d\tilde{f}(y)}{dy}. \tag{4.34}$$

Expansions of these functions for small y are found to be

$$\tilde{f}(y) = 0{\cdot}2009 I_1 - 0{\cdot}0667 I_2 y^{\frac{2}{3}} + \dots, \tag{4.35}$$

$$\tilde{g}(y) = 0{\cdot}3348 I_1 - 0{\cdot}1333 I_2 y^{\frac{2}{3}} + \dots, \tag{4.36}$$

where

$$I_1 = \int_0^\infty x^{-\frac{1}{3}} R(x)\, dx, \tag{4.37}$$

$$I_2 = \int_0^\infty x^{\frac{1}{3}} R(x)\, dx. \tag{4{\cdot}38}$$

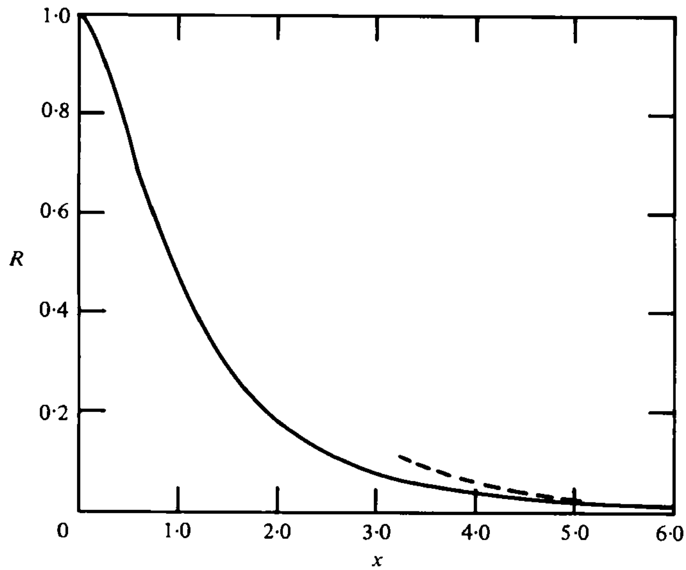

FIGURE 1. The inertial range function R versus the dimensionless wavenumber $x = k\epsilon^{\frac{1}{2}}(t-s)^{\frac{3}{2}}$. The dashed line is $R = 0{\cdot}012(x/6)^{-\frac{11}{3}}$.

For large y

$$\tilde{f}(y) = 0{\cdot}6575y^{-\frac{2}{3}} + \ldots, \tag{4.39}$$

$$\tilde{g}(y) = 0{\cdot}8767y^{-\frac{2}{3}} + \ldots. \tag{4.40}$$

Equation (4.32) has been solved numerically by a method similar to that described by Kraichnan (1966*a*). Basically the integral is expressed in terms of a finite number of function values of R, $R_i = R(hi), i = 1, 2, \ldots, N$, and the derivative approximated by a difference. Thus

$$R_{n+1} = R_n + \tfrac{1}{2}h[F_{n+1}(R_1, \ldots, R_N) + F_n(R_1, \ldots, R_N)], \tag{4.41}$$

where F_n is the integral. If the F_n's are evaluated by an initial set of R_n's and a new set computed by the above formula this is found to be a divergent scheme. However, if the F_n's are always evaluated using the most current values of the R_n's the iteration process converges. This iteration process is

$$R_{n+1}^{(j+1)} = R_n^{(j+1)} + \tfrac{1}{2}h[F_{n+1}(R_1^{(j+1)}, \ldots, R_n^{(j+1)}, R_{n+1}^{(j)}, \ldots, R_N^{(j)}) + F_n(R_1^{(j+1)}, \ldots, R_n^{(j+1)}, R_{n+1}^{(j)}, \ldots, R_N^{(j)})], \tag{4.42}$$

where $R^{(j)}$ refers to the jth iteration. In approximating the integral use was made of the asymptotic approximation $R(x) = R_N(x/Nh)^{-\frac{11}{3}}$ whenever u or v was greater than Nh. This was done numerically when $u < 2Nh$ and analytically when $u \geqslant 2Nh$.

The basic result of this computation is the function $R(x)$ which is shown in figure 1 together with the $x^{-\frac{11}{3}}$ asymptote. This function corresponds to the function called R by Kraichnan (1966*a*), computed by the abridged LHDI approximation, except that his argument is $s = \epsilon^{\frac{1}{3}}k^{\frac{2}{3}}(t-s)$ (a dimensionless time) while in the present paper it is $x = \tilde{\epsilon}^{\frac{1}{2}}k(t-s)^{\frac{3}{2}}$ (a dimensionless wavenumber). The functions appear to be similar except for a considerably longer tail on Kraichnan's function, which behaves like

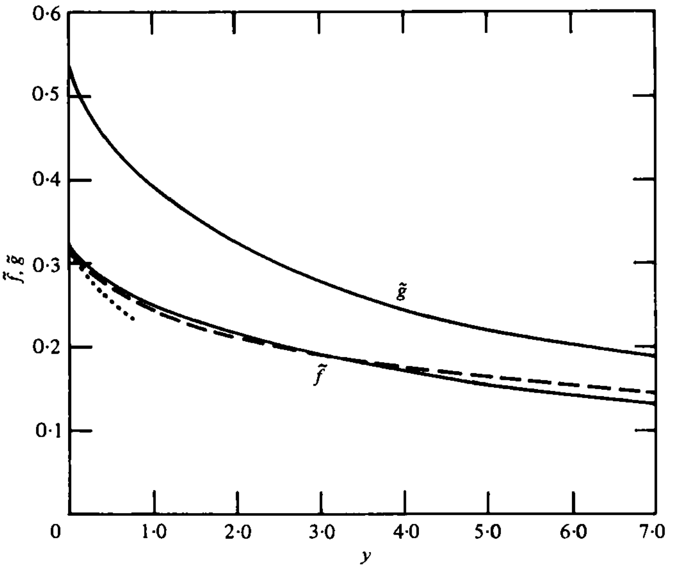

FIGURE 2. The dimensionless longitudinal and latitudinal turbulent pair diffusivities in the inertial range versus the dimensionless distance $y = \rho\tilde{\epsilon}^{-\frac{1}{2}}(t-s)^{-\frac{3}{2}}$. The dotted line is the asymptotic result $\tilde{f} = 0{\cdot}321 - 0{\cdot}105y^{\frac{2}{3}}$. The dashed line is the approximation $\tilde{f} = 0{\cdot}32/(1+0{\cdot}33y^{\frac{2}{3}})$.

$x^{-\frac{4}{3}}$ compared with the present $x^{-\frac{11}{3}}$. The longer tail accounts for different estimates of the value of the integral I_1. In the present work

$$I_1 = 1{\cdot}60, \tag{4.43}$$

$$I_2 = 1{\cdot}57, \tag{4.44}$$

while Kraichnan computes

$$I_R = \int_0^\infty R_K(s)\,d\bar{s} = \tfrac{2}{3}C^{\frac{1}{4}}I_1,$$

obtaining $I_R = 1{\cdot}87$. With $C = 1{\cdot}77$ this corresponds to $I_1 = 2{\cdot}43$, considerably larger. The values of I_1 and I_2 given above make

$$\tilde{f} = 0{\cdot}321 - 0{\cdot}105y^{\frac{2}{3}} + \ldots, \tag{4.45}$$

$$\tilde{g} = 0{\cdot}535 - 0{\cdot}210y^{\frac{2}{3}} + \ldots. \tag{4.46}$$

The functions $\tilde{f}$ and $\tilde{g}$ have been calculated numerically from (4.33), evaluating the trigonometric functions recursively. The results are shown in figure 2 together with the asymptotic results. These functions correspond to Kraichnan's $J_{\parallel}$ and $J_{\times}$ except for multiplying constants and arguments of the functions. Whereas the present analysis gives $\tilde{f}(0) = 0{\cdot}331$, Kraichnan's value of this number would be $0{\cdot}752$. This comparison will be pursued in § 6.

The dotted line in figure 2 is the function

$$\tilde{f}(y) = \frac{0{\cdot}32}{1+0{\cdot}33y^{\frac{2}{3}}}, \tag{4.47}$$

which is seen to approximate $\tilde{f}$ adequately.

5. Pair dispersion and the pair distribution function

Once the turbulent pair diffusivity has been determined by the methods of the last few sections, equation (2.14) provides a linear partial differential equation for the pair distribution function $P_r(\mathbf{e}, t|\mathbf{q}, s)$. In the subsections below, the special cases where the pair diffusivity has been determined in § 4 will be examined. In none of these cases is P_r itself found. However, it is possible to determine the distribution function for the magnitude of the pair separation.

5.1. *The dissipation range*

When the particles are close enough together that they only feel dissipation range eddies, P_r should be a solution of equation (2.14) with $\mathbf{D}$ given by (4.8). For convenience define a new time variable by

$$T = \int_0^\tau 2f_0(\tau)\, d\tau = \tfrac{1}{3}\ln\cosh(\tau/\sqrt{5}), \tag{5.1}$$

this function behaves like $\tau^2/30$ as $\tau \to 0$ and like $\tau/3\sqrt{5}$ as $\tau \to \infty$. Then the equation for P_r becomes

$$\frac{\partial P_r(\boldsymbol{\rho}, t|\mathbf{q}, 0)}{\partial T} = (2\rho^2\mathbf{I} - \boldsymbol{\rho}\boldsymbol{\rho}) : \frac{\partial}{\partial \boldsymbol{\rho}}\frac{\partial}{\partial \boldsymbol{\rho}} P_r(\boldsymbol{\rho}, t|\mathbf{q}, 0), \tag{5.2}$$

$$P_r = \delta(\boldsymbol{\rho} - \mathbf{q}) \quad \text{at} \quad T = 0,$$

where $\mathbf{q}$ is a fixed non-statistical initial separation vector.

Since the right-hand side of this equation is homogeneous in $\boldsymbol{\rho}$, independent moment equations of all orders may be computed, by integrating by parts. Using the notation

$$\langle \rho_i \rangle = \int \rho_i P_r\, d\boldsymbol{\rho},$$

$$\langle \rho_i \rho_j \rangle = \int \rho_i \rho_j P_r\, d\boldsymbol{\rho},$$

one readily finds

$$\frac{\partial}{\partial T}\langle \rho_i \rangle = 0, \quad \langle \rho_i \rangle = q_i \quad \text{at} \quad T = 0, \tag{5.3}$$

$$\frac{\partial}{\partial T}\langle \rho_i \rho_j \rangle = 4\langle \rho^2 \rangle \delta_{ij} - 2\langle \rho_i \rho_j \rangle. \tag{5.4}$$

Equation (5.3) shows $\langle \boldsymbol{\rho} \rangle = \mathbf{q}$ for all time. Equation (5.4) may first be solved for $\langle \rho^2 \rangle$, giving

$$\langle \rho^2 \rangle = q^2 e^{10T} \tag{5.5}$$

and then for $\langle \rho_i \rho_j \rangle - \frac{1}{3}\langle \rho^2 \rangle \delta_{ij}$, giving the final result

$$\langle \rho_i \rho_j \rangle = \tfrac{1}{3}q^2\delta_{ij}\, e^{10T} + (q_i q_j - \tfrac{1}{3}q^2\delta_{ij})\, e^{-2T}. \tag{5.6}$$

As T increases $\langle \rho_i \rho_j \rangle$ rapidly becomes isotropic, forgetting its dependence on the direction of the initial separation vector.

It is apparent that these solutions are only physically valid for a limited time. If the initial separation q is in the dissipation range, equation (5.5) shows that eventually the particles will be affected by eddies outside of this range.

It is possible to solve for the distribution function for the magnitude of the particle separations. This function is defined by $\langle \delta(R - \rho) \rangle$, that is, by

$$P(R, t|q, 0) = \int \delta(R - \rho)\, P_r(\boldsymbol{\rho}, t|\mathbf{q}, 0)\, d\boldsymbol{\rho}. \tag{5.7}$$

It has the normalization

$$\int_0^\infty P\,dR = 1.$$

An equation for P may be derived as follows. First differentiate equation (5.7) with respect to T, obtaining

$$\frac{\partial P}{\partial T} = \int \delta(R-\rho)\frac{\partial P_r}{\partial T}d\boldsymbol{\rho}, \tag{5.8}$$

then substitute for $\partial P_r/\partial T$ from (5.2). Integrate by parts twice to shift the derivatives onto the delta function and then from derivatives with respect to ρ to derivatives with respect to R, which can be taken outside of the integral. This gives the result

$$\frac{\partial P}{\partial T} = R\frac{\partial}{\partial R}R\frac{\partial P}{\partial R} - R\frac{\partial P}{\partial R} - 2P \tag{5.9}$$

to be solved with the initial condition

$$P = \delta(R-q) \quad \text{at} \quad T = 0.$$

This can be reduced to the heat equation by the change of variables $X = \ln R/q - T$, giving

$$\frac{\partial}{\partial T}Pe^{2T} = \frac{\partial^2}{\partial X^2}Pe^{2T} \tag{5.10}$$

with initial condition

$$Pe^{2T} = \frac{1}{q}\delta(X) \quad \text{at} \quad T = 0.$$

The well-known point-source solution of the heat equation may be manipulated to give the final form

$$P(R,t|q,0) = \frac{1}{R(4\pi T)^{\frac{1}{2}}}\exp\left\{-\frac{(\ln R/q - 3T)^2}{4T}\right\}, \tag{5.11}$$

which is the standard form of the log-normal distribution. That is, $\ln R/q$ is distributed normally with mean $3T$ and variance $(2T)^{\frac{1}{2}}$.

The function P may also be derived as $4\pi\rho^2 P_r''$ where P_r'' is the spherically symmetric solution of (5.2) which satisfies the initial condition $\delta(\rho-q)/4\pi q^2$.

These results are in substantial agreement with those of other investigators. Batchelor (1952) predicted exponential growth of the particle separation, including the $(\epsilon/\nu)^{\frac{1}{2}}$ scaling of the time. Kraichnan (1974) obtained both exponential growth and the log-normal distribution by making some statistical assumptions.

5.2. *The two-dimensional inertial range*

In this case $\mathbf{D}$ is given by (4.19) and the equation for P_r is

$$\left.\begin{aligned} \frac{\partial P_r(\boldsymbol{\rho},t|\mathbf{q},0)}{\partial T_1} &= (3\rho^2\mathbf{I} - 2\boldsymbol{\rho}\boldsymbol{\rho}):\frac{\partial}{\partial\boldsymbol{\rho}}\frac{\partial}{\partial\boldsymbol{\rho}}P_r(\boldsymbol{\rho},t|\mathbf{q},0),\\ P_r &= \delta(\boldsymbol{\rho}-\mathbf{q}) \quad \text{at} \quad T_1 = 0, \end{aligned}\right\} \tag{5.12}$$

where

$$T_1 = \int_0^{\tau_1} 2f_1(\tau_1)\,d\tau_1 = \tfrac{1}{2}\ln\cosh(2\tau_1) \tag{5.13}$$

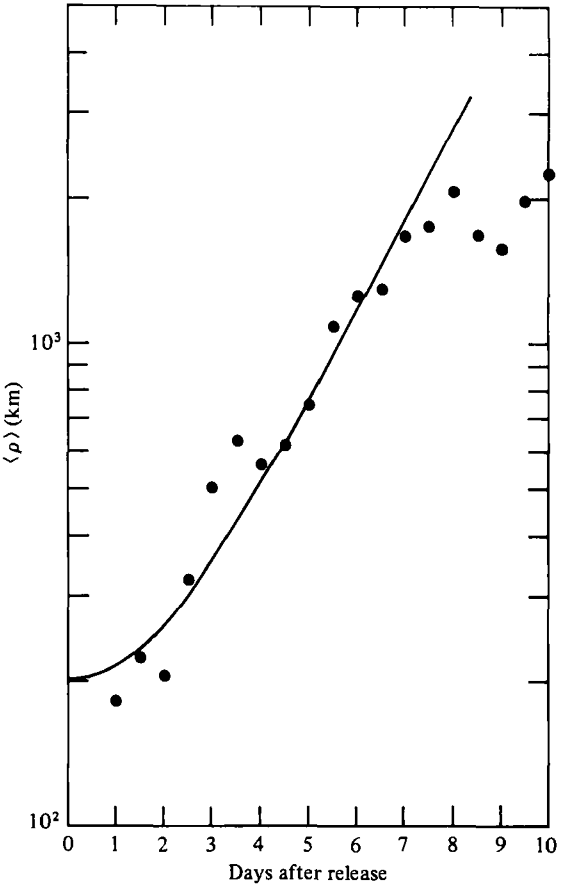

FIGURE 3. The mean separation of balloon pairs versus the number of days after release. The solid curve is calculated from (5.19). The solid points are due to Er-El & Peskin, reduced from TWERLE data.

goes like τ_1^2 for small τ_1 and like τ_1 for large τ_1 and $\boldsymbol{\rho}$ is understood to be a two-dimensional vector. As in the dissipation range, homogeneity of the right-hand side of (5.12) ensures independent moment equations. In particular the second moment equation,

$$\left.\begin{aligned} \frac{d}{dT_1}\langle\rho_i\rho_j\rangle &= 6\langle\rho^2\rangle\delta_{ij} - 4\langle\rho_i\rho_j\rangle, \\ \langle\rho_i\rho_j\rangle &= q_i q_j \quad \text{at} \quad T_1 = 0, \end{aligned}\right\} \tag{5.14}$$

has solution

$$\langle\rho^2\rangle = q^2 e^{8T_1}, \tag{5.15}$$

$$\langle\rho_i\rho_j\rangle = \tfrac{1}{2}q^2 e^{8T_1}\delta_{ij} + (q_i q_j - \tfrac{1}{2}q^2\delta_{ij})\, e^{-4T_1}. \tag{5.16}$$

The distribution function for the magnitude of particle separations is defined as before by (5.7) and an equation for this quantity may be derived from (5.12) by the same procedure, which gives

$$\frac{\partial P}{\partial T_1} = R\frac{\partial}{\partial R}R\frac{\partial P}{\partial R} - P \tag{5.17}$$

with initial condition, $P = \delta(R-q)$ at $T_1 = 0$. The solution is

$$P(R,t|q,0) = \frac{1}{R(4\pi T_1)^{\frac{1}{2}}}\exp\left\{-\frac{(\ln R/q-2T_1)^2}{4T_1}\right\}, \tag{5.18}$$

a log-normal distribution again but with slightly different numbers.

Independent moment equations may also be obtained from (5.17). In particular,

$$\langle \rho \rangle = qe^{3T_1}. \tag{5.19}$$

There have been two large-scale balloon release experiments in which groups of constant pressure balloons were tracked by satellite over a period of several months. The Eole experiment (Morel & Bandeen 1973; Morel & Larcheveque 1974) was launched in 1971 and the TWERLE experiment (Julian *et al.* 1977) in 1975. In both experiments pair separation statistics have been reported, the TWERLE data having been recently studied by Er-El & Peskin (1979, private communication). They are notably non-isotropic at large times after release, with preferred separation in the zonal direction. For times shorter than 6 or 7 days after release this effect is smaller but still larger than can be explained by zonal bias in the initial separation vector which results in the effects seen in (5.16). Some of the early time TWERLE results are shown in figure 3, where measured values of $\langle \rho \rangle$ are plotted versus time after release from the ground and compared with (5.19) in which T_1 is given by (5.13) with $\tau_1 = t/a$ and a and q are 6·5 days and 200 km respectively. The agreement is satisfactory, though it takes quite a few days to reach the asymptote $\langle \rho \rangle \sim \exp(3t/a)$, which would be a straight line on this figure. Similar results could be shown for the EOLE data.

Lin (1971) has also predicted exponential separation.

5.3. *The three-dimensional inertial range*

In the three-dimensional inertial range P_r should satisfy (2.14) with $\mathbf{D}$ given by (3.1) with f in the special form (4.33). This equation is not simple enough to calculate independent moment equations; however, it is possible to derive an equation for the distribution function for the magnitude of particle separations. The same procedure described in § 5.1 gives

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial R}2f\left(\frac{\partial P}{\partial R}-\frac{2}{R}P\right)$$

or

$$\frac{\partial P/R^2}{\partial t} = \frac{1}{R^2}\frac{\partial}{\partial R}2R^2 f\frac{\partial P/R^2}{\partial R}, \tag{5.20}$$

where f is given by

$$f = \tilde{\epsilon}^{\frac{1}{3}}R^{\frac{4}{3}}\tilde{f}(Y), \quad Y = R/\tilde{\epsilon}^{\frac{1}{2}}t^{\frac{3}{2}} \tag{5.21}$$

and the initial condition is $P(R,0|q,0)$. The solution satisfying this initial condition is not known; however there is a similarity solution due to Kraichnan (1966*b*) which generalizes the results of Richardson (1926) and Roberts (1961). This solution is valid for time large enough that $\tilde{\epsilon}^{\frac{1}{2}}t^{\frac{3}{2}} \gg q$ and unlike the solutions in the last two subsections becomes independent of q in this limit. Because of the normalization

$$\int_0^\infty P\,dR = 1,$$

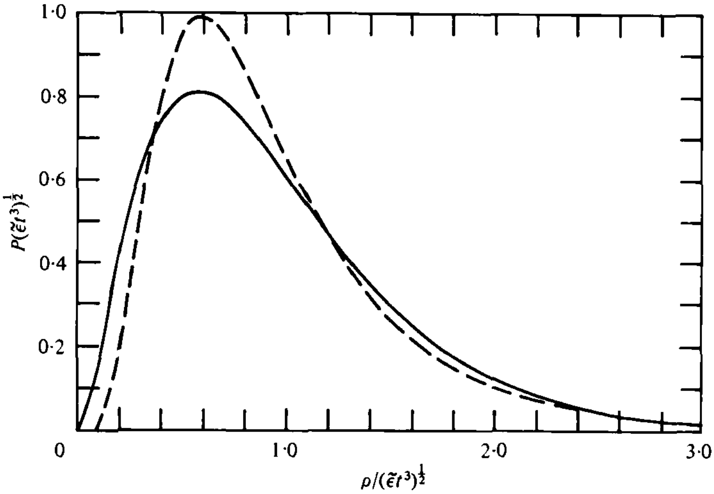

FIGURE 4. The probability density for the separation distance between particle pairs in inertial range turbulence at large time after release. The dashed line is a log-normal distribution with the same mean and variance.

one seeks a similarity solution in the form

$$P(R,t|q,0) = (\tilde{\epsilon}^{\frac{1}{2}}t^{\frac{3}{2}})^{-1}\tilde{P}(Y) \tag{5.22}$$

with the property

$$\int_0^\infty \tilde{P}(Y)\,dY = 1.$$

The function $\tilde{P}$ must satisfy

$$-\frac{3}{2}\frac{d}{dY}\,Y\tilde{P} = \frac{d}{dY}2Y^{\frac{4}{3}}\tilde{f}(Y)\frac{d\tilde{P}}{dY}-\frac{2\tilde{P}}{Y}\Big) \tag{5.23}$$

and the solution is easily found to be

$$\tilde{P}(Y) = NY^2\exp\left\{-\frac{3}{4}\int_0^Y \frac{dy}{y^{\frac{1}{3}}\tilde{f}(y)}\right\}, \tag{5.24}$$

where

$$N = \left(\int_0^\infty Y^2\exp\left\{-\frac{3}{4}\int_0^Y \frac{dy}{y^{\frac{1}{3}}\tilde{f}(y)}\right\}dy\right)^{-1}. \tag{5.25}$$

One should note that the solution is independent of the specific form of the function $\tilde{f}$. Also one should note that Kraichnan's (1966*b*) solution is in error; the integral in the exponent in his equation (5.4) is missing.

Computations have been made using (4.47) as an approximation to $\tilde{f}$. This gives

$$\tilde{P}(Y) = NY^2\exp\{-3{\cdot}52(Y^{\frac{2}{3}}+0{\cdot}165\,Y^{\frac{4}{3}})\} \tag{5.26}$$

with $N = 37{\cdot}0$. This is plotted in figure 4. The maximum of this function occurs at $Y = 0{\cdot}57$, which means that the most probable separation is

$$\rho_m = 0{\cdot}57(\tilde{\epsilon}t^3)^{\frac{1}{2}} = 0{\cdot}87(\epsilon t^3)^{\frac{1}{2}}.$$

Also plotted is a log-normal distribution with the same mean and variance.

The first few moments have been calculated, with the results $\langle Y\rangle = 0{\cdot}97$, $\langle Y^2\rangle = 1{\cdot}30$, $\langle Y^3\rangle = 2{\cdot}26$, $\langle Y^4\rangle = 4{\cdot}80$, $\langle Y^5\rangle = 12{\cdot}07$, $\langle Y^6\rangle = 34{\cdot}92$. The corresponding moments for the log-normal distribution are respectively 0·97, 1·30, 2·43, 6·36, 23·11, 116·9 which rapidly become larger because of the longer tail on the log-normal distribution.

From the second moment one finds

$$\langle \rho^2 \rangle = 1{\cdot}30\tilde{\epsilon}t^3 = 3{\cdot}06\epsilon t^3, \tag{5.27}$$

which is Richardson's (1926) law. Kraichnan (1966*b*) gives $\langle \rho^2 \rangle = 2{\cdot}42\epsilon t^3$. Experiments cited by Monin & Yaglom (1975) verify the exponent in this law but the coefficient has not been established because of the difficulty in measuring ϵ.

From the above moments one can calculate $\langle \rho^4 \rangle / \langle \rho^2 \rangle^2 = 2{\cdot}83$ and $\langle \rho^6 \rangle / \langle \rho^2 \rangle^3 = 15{\cdot}8$ which can be compared with the values 2·27 and 8·86 calculated by Kraichnan.

6. Turbulent diffusion

Let $c(\mathbf{x}, t)$ be the concentration of a passive scalar satisfying

$$\frac{\partial c}{\partial t} + \mathbf{u} . \nabla c = \mathscr{D} \nabla^2 c, \tag{6.1}$$

where $\mathscr{D}$ is a molecular diffusivity. Suppose, to begin, that $\mathscr{D}$ is zero. In this case c is simply carried with the fluid particles and

$$c(\mathbf{x}, t) = c(\mathbf{R}(\mathbf{x}, t|0), 0). \tag{6.2}$$

Using this result, one finds

$$\langle c(\mathbf{x}_1, t)\, c(\mathbf{x}_2, t) \rangle$$
$$= \int \langle c(\mathbf{X}_1, 0)\, c(\mathbf{X}_2, 0)\, \delta(\mathbf{X}_1 - R(\mathbf{x}_1, t|0))\, \delta(\mathbf{X}_2 - \mathbf{R}(\mathbf{x}_2, t|0)) \rangle\, d\mathbf{X}_1\, d\mathbf{X}_2 \tag{6.3}$$
$$= \int \langle c(\mathbf{X}_1, 0)\, c(\mathbf{X}_2, 0) \rangle\, P_2(\mathbf{x}_1, \mathbf{x}_2, t | \mathbf{X}_1, \mathbf{X}_2, 0)\, d\mathbf{X}_1\, d\mathbf{X}_2, \tag{6.4}$$

where the last line follows because the scalar is passive, i.e. does not affect the velocity field. For a homogeneous scalar field, the scalar correlation function is a function only of the separation of the two points,

$$\langle c(\mathbf{x}_1, t)\, c(\mathbf{x}_1 + \boldsymbol{\rho}, t) \rangle = R_c(\boldsymbol{\rho}, t) \tag{6.5}$$

and (6.4) may be written

$$R_c(\boldsymbol{\rho}, t) = \int R_c(\mathbf{q}, 0)\, P_r(\boldsymbol{\rho}, t | \mathbf{q}, 0)\, d\mathbf{q}. \tag{6.6}$$

Therefore $\mathbf{R}_c$ satisfies the same equation as P_r, namely

$$\frac{\partial \mathbf{R}_c}{\partial t} = \frac{\partial}{\partial \boldsymbol{\rho}} . 2\mathbf{D} . \frac{\partial \mathbf{R}_c}{\partial \boldsymbol{\rho}}. \tag{6.7}$$

It is seen that in the absence of molecular diffusivity the effective turbulent diffusivity is the turbulent pair diffusivity.

If molecular diffusivity is not neglected an equation for $\mathbf{R}_c$ may be derived from (6.1),

$$\frac{\partial R_c}{\partial t} = -\frac{\partial}{\partial \boldsymbol{\rho}} . \langle (\mathbf{u}(\mathbf{x}_2, t) - \mathbf{u}(\mathbf{x}_1, t))\, c(\mathbf{x}_1, t)\, c(\mathbf{x}_2, t) \rangle + 2\mathscr{D} \frac{\partial}{\partial \boldsymbol{\rho}} . \frac{\partial}{\partial \boldsymbol{\rho}} R_c. \tag{6.8}$$

If one could reason that the flux term were independent of the molecular diffusivity then the appropriate equation would be

$$\frac{\partial R_c}{\partial t} = \frac{\partial}{\partial \boldsymbol{\rho}} . 2\mathbf{D} . \frac{\partial R_c}{\partial \boldsymbol{\rho}} + 2\mathscr{D} \frac{\partial}{\partial \boldsymbol{\rho}} . \frac{\partial R_c}{\partial \boldsymbol{\rho}}. \tag{6.9}$$

It is known, however, from the work of Batchelor (1959) and Kraichnan (1968) that this is not a uniformly valid approximation at all scales. One can reason that the approximation is good if the diffusive relaxation time, $(\mathscr{D}k^2)^{-1}$, is large compared with the eddy turnaround time for eddies of the same size, k^{-1}, as the scalar inhomogeneity under consideration. In the inertial range where the turnaround time is $(\epsilon k^2)^{-\frac{1}{3}}$ this requires

$$k \ll (\epsilon/\mathscr{D}^3)^{\frac{1}{4}} \tag{6.10}$$

while in the dissipation range where the turnaround time is $(\nu/\epsilon)^{\frac{1}{2}}$ it requires

$$k \ll (\epsilon/\nu\mathscr{D}^2)^{\frac{1}{4}}. \tag{6.11}$$

Therefore, equation (6.9) is in error at large wavenumbers or at small values of ρ.

In the following, some results on the asymptotic form of the scalar spectrum will be obtained. Equation (6.9) will be used for analytical reasons, even though it is not completely justified. The equation gives the correct scaling with diffusivity and even gives results which agree moderately well with experiment in the diffusion dominated limit.

For homogeneous turbulence $\langle c \rangle$ is constant, and without loss of generality may be taken to be zero. In this case $R_c(\rho, t)$ will tend to zero as $\rho \to \infty$ and one can define

$$\Phi_c(k,t) = \int e^{-i\mathbf{k}\cdot\boldsymbol{\rho}} R_c(\rho,t)\, d\boldsymbol{\rho}. \tag{6.12}$$

From the inverse transform of this, one finds

$$\langle c^2 \rangle = R_c(0,t) = \int_0^\infty G(k,t)\, dk, \tag{6.13}$$

where the scalar spectrum function $G(k,t)$ is related to Φ_c and R_c by

$$G(k,t) = k^2 \Phi_c(k,t)/2\pi^2 = \frac{2}{\pi} k \frac{\partial}{\partial k} \frac{1}{k} \int_0^\infty \frac{\partial R_c(\rho,t)}{\partial \rho} \sin k\rho \, d\rho. \tag{6.14}$$

For isotropic homogeneous turbulence, equation (6.9) becomes

$$\frac{\partial R_c(\rho,t)}{\partial t} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} 2\rho^2 (f+\mathscr{D}) \frac{\partial R_c(\rho,t)}{\partial \rho}. \tag{6.15}$$

In order to obtain the high-wavenumber behaviour of the scalar spectrum the small-ρ behaviour of R_c is required. For small ρ one looks for solutions of form

$$R_c(\rho,t) = \langle c^2 \rangle + R_c'(\rho,t),$$

where $R_c' \to 0$ as $\rho \to 0$. For sufficiently small ρ, one can write

$$K = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} 2\rho^2 (f+\mathscr{D}) \frac{\partial R_c'}{\partial \rho} \tag{6.16}$$

with $K = d\langle c^2 \rangle/dt$. This may be integrated to give

$$\frac{\partial R_c'}{\partial \rho} = \frac{1}{6} \frac{K\rho}{f+\mathscr{D}} \tag{6.17}$$

or

$$R_c' = \frac{K}{6} \int_0^\rho \frac{\rho\, d\rho}{f+\mathscr{D}}. \tag{6.18}$$

When equation (6.17) is substituted into (6.14) the asymptotic form of $G(k,t)$ may be found from

$$G(k,t) = \frac{2}{\pi}\frac{K}{6}k\frac{\partial}{\partial k}\frac{1}{k}\int_0^\infty \frac{\rho}{f(\rho,t)+\mathscr{D}}\sin k\rho\, d\rho. \tag{6.19}$$

6.1. *The dissipation range*

When $\rho \ll (\nu^3/\epsilon)^{\frac{1}{4}}$, f is given by (4.2), $f = (\epsilon\nu)^{\frac{1}{2}}\rho^2 f_0(\tau)$, where $f_0 \to 1/6\sqrt{5}$ as $\tau \to \infty$. In this case equation (6.18) may be integrated to give

$$R_c' = \frac{1}{12}\frac{K}{(\epsilon/\nu)^{\frac{1}{2}}f_0}\ln\{1+(\epsilon/\nu\mathscr{D}^2)^{\frac{1}{2}}f_0\rho^2\}. \tag{6.20}$$

When $\rho \gg (\epsilon/\nu\mathscr{D}^2)^{\frac{1}{4}}$, so that inequality (6.11) is satisfied (in order to also be in the dissipation range this requires $\mathscr{D} \ll \nu$) this reduces to Batchelor's (1959) form

$$R_c' = \frac{1}{12}\frac{K}{(\epsilon/\nu)^{\frac{1}{2}}f_0}\ln\{(\epsilon/v\mathscr{D}^2)^{\frac{1}{2}}f_0\rho^2\} \tag{6.21}$$

with some slightly different coefficients which will be discussed below. It is interesting to note that when $\rho \ll (\epsilon/\nu\mathscr{D}^2)^{\frac{1}{4}}$ which completely violates inequality (6.11), (6.20) reduces to the exact result

$$R_c' = \frac{1}{12}\frac{K}{\mathscr{D}}\rho^2. \tag{6.22}$$

Equation (6.19) may be integrated to give

$$G(k,t) = -\frac{K\sqrt{5}}{(\epsilon/\nu)^{\frac{1}{2}}}\frac{1}{k}(1+ka)\,e^{-ka}, \tag{6.23}$$

where

$$a = \left(\frac{\mathscr{D}^2\nu}{\epsilon}\right)^{\frac{1}{4}} 6^{\frac{1}{2}}5^{\frac{1}{4}},$$

and the asymptotic value of f_0 has been used. When inequality (6.11) is employed this reduces to

$$G(k,t) = \frac{K\sqrt{5}}{(\epsilon/\nu)^{\frac{1}{2}}}\frac{1}{k}, \tag{6.24}$$

which is Batchelor's (1959) viscous–convective result with a factor $\sqrt{5}$ instead of 2. Experiments of Gibson & Schwarz (1963) are in satisfactory agreement with either of these numbers. Furthermore the result (6.23) when extended into the questionable diffusive range is also in satisfactory agreement with these experiments (and with the quite different analytical expression of Batchelor's) over the range of the experiments.

The value $\sqrt{5}$ in equation (6.24) is inversely proportional to f_0. The agreement with experiment cited above tends to support the use of (2.25) in the dissipation range.

6.2. *The inertial range*

In high-Reynolds-number turbulence where there is an extensive inertial range, the appropriate form for f is given by (4.25),

$$f = \tilde{\epsilon}^{\frac{1}{3}}\rho^{\frac{4}{3}}\tilde{f}(y), \quad y = \rho/\tilde{\epsilon}^{\frac{1}{2}}t^{\frac{3}{2}}. \tag{6.25}$$

For sufficiently large t, $\tilde{f}$ may be replaced by $\tilde{f}_0 = 0{\cdot}321$. In this case the expression (6.19) for the scalar spectrum may be written

$$G(k,t) = \frac{K}{3\pi\tilde{\epsilon}^{\frac{1}{3}}\tilde{f}_0}\,k\frac{\partial}{\partial k}\,k^{-\frac{5}{3}}\int_0^\infty \frac{x\sin x}{b+x^{\frac{4}{3}}}\,dx, \tag{6.26}$$

where

$$b = \frac{1}{\tilde{f}_0}\left[\frac{k}{(\tilde{\epsilon}/\mathscr{D}^3)^{\frac{1}{4}}}\right]^{\frac{4}{3}}.$$

In the inertial–convective limit, where inequality (6.10) is satisfied, the parameter $b \to 0$ and (6.26) gives

$$G(k,t) = -\beta_3\frac{K}{\epsilon^{\frac{1}{3}}}\,k^{-\frac{5}{3}}, \tag{6.27}$$

where

$$\begin{aligned}\beta_3 &= \frac{1}{\pi}\frac{5\sqrt{3}}{12}\,\Gamma(\tfrac{5}{3})\,C^{-\frac{1}{2}}\tilde{f}_0^{-1} \\ &= 0{\cdot}486\end{aligned} \tag{6.28}$$

upon using $C = 1{\cdot}77$ and $\tilde{f}_0 = 0{\cdot}321$. This form for the scalar spectrum is due to Obukhov (1949) and Corrsin (1951) and β_3 is called the Obukhov–Corrsin number. Kraichnan (1966*b*) gives $\beta_3 = 0{\cdot}208$ (which results from (6.28) with $C = 1{\cdot}77$ and $\tilde{f}_0 = 0{\cdot}752$).

Experimental values of β_3 have been obtained by a number of investigators. These results are collected in table 1. The value $\beta_3 = 0{\cdot}49$ of the present investigation appears to be within the experimental scatter of most of these.

In the inertial–convective limit equation (6.18) gives

$$R_c' = \frac{1}{4}\frac{K}{\tilde{\epsilon}^{\frac{1}{3}}\tilde{f}_0}\,\rho^{\frac{2}{3}}. \tag{6.29}$$

In the inertial–diffusive limit where inequality (6.10) is violated, an asymptotic expansion of (6.26) for large b, gives

$$G(k,t) = \frac{3^{\frac{1}{2}}\,13\Gamma(10/3)}{18\pi}\,\frac{K\tilde{f}_0\tilde{\epsilon}^{\frac{1}{3}}}{\mathscr{D}^2}\,k^{-\frac{13}{3}}. \tag{6.30}$$

This is not in agreement with the generally accepted $k^{-\frac{17}{3}}$ behaviour predicted by Batchelor, Howells & Townsend (1959) which takes into account that the molecular diffusivity can modify the turbulent diffusivity. This inertial–diffusive limit can only occur in fluids for which $\mathscr{D} \gg \nu$ since $k \ll (\epsilon/\nu^3)^{\frac{1}{4}}$ to be in the inertial range and $k \gg (\epsilon/\mathscr{D}^3)^{\frac{1}{4}}$ to be in the diffusive range. There have been experiments by Granatstein, Buchsbaum & Bugnolo (1966) in a slightly ionized plasma ($\nu/\mathscr{D} = 0{\cdot}07$) and by Clay (1973) in mercury ($\nu/\mathscr{D} = 0{\cdot}02$) which appear to support the $\frac{17}{3}$ law.

7. Decay of the scalar variance or fluctuation intensity

The spectral results of the last section were expressed in terms of $d\langle c^2\rangle/dt$ which is not known. The fluctuation intensity $\langle c^2\rangle$ is a quantity one would like to predict because it is a measure of the degree of mixing of the scalar pollutant or chemical species. One can imagine a situation in which the scalar is originally distributed in some uniformly random lumpy manner (with $\langle c\rangle = 0$) specified by R_c or G. In the

Reference	Value of $\beta_3 \pm$ standard deviation
Gibson & Schwarz (1963)	$0{\cdot}58 \pm 0{\cdot}08$
Grant *et al.* (1968)	$0{\cdot}52 \pm 0{\cdot}10$
Paquin & Pond (1971)	$0{\cdot}41 \pm 0{\cdot}07$ (temperature)
	$0{\cdot}40 \pm 0{\cdot}09$ (humidity)
Williams (1974)	$0{\cdot}46 \pm 0{\cdot}02$ (estimated from Champagne *et al.* 1977)
Champagne *et al.* (1977)	$0{\cdot}41 \pm 0{\cdot}02$

TABLE 1. Experimental values of the Obukhov–Corrsin number.

absence of molecular diffusivity $\langle c^2 \rangle$ will remain constant while, according to Batchelor (1952), $\langle \nabla c . \nabla c \rangle$ will continue to increase with time because of the stretching, and consequent squeezing together, of neighbouring surfaces of constant c. This means that the spectrum $G(k,t)$ will evolve in such a way that its integral remains constant, while the integral of $k^2 G$ increases with time, thus spreading G to higher wavenumbers. Ultimately molecular diffusivity will become important and $\langle c^2 \rangle$ will decay. It will be shown that when the diffusion is controlled by inertial range eddies $\langle c^2 \rangle$ will finally decay at a rate independent of the Schmidt number, $S = \nu/\mathscr{D}$, and consequently independent of the molecular diffusivity.

The case will be considered in which the initial scale of the scalar spectrum is much smaller than the energy scale of the turbulence and the diffusion is dominated by the inertial subrange of the energy spectrum. One can envision experiments similar to those of Warhaft & Lumley (1978) in which the scalar is temperature introduced by a heated wire array downstream of a turbulence-producing grid. In the case under consideration it is required that the heated wire spacing be in the inertial range of the turbulence.

The basic equation for $R_c(\rho, t)$ is equation (6.15) which is to be solved with initial value $R_c(\rho, 0)$ specified. It will be assumed that $R_c(\rho, 0)$ scales in the inertial range. The solution will be obtained by matching an inner expansion, similar to (6.18), to an outer expansion derived from the similarity solution (5.24).

It will be assumed that f has a composite form which covers both the inertial and dissipation ranges,

$$f = \tilde{\epsilon}^{\frac{1}{3}} \rho^{\frac{4}{3}} \tilde{f}(\rho/\tilde{\epsilon}^{\frac{1}{2}} t^{\frac{3}{2}}) + \nu F(\rho/(\nu^3/\tilde{\epsilon})^{\frac{1}{4}}, (\epsilon/\nu)^{\frac{1}{2}} t), \tag{7.1}$$

where the function F is not precisely known, but has the properties

$$F \to 0 \quad \text{as} \quad \rho/(\nu^3/\tilde{\epsilon})^{\frac{1}{4}} \to \infty \tag{7.2}$$

and

$$F \sim [\rho/(\nu^3/\epsilon)^{\frac{1}{4}}]^2 f_0((\epsilon/\nu)^{\frac{1}{2}} t) - [\rho/(\nu^3/\tilde{\epsilon})^{\frac{1}{4}}]^{\frac{4}{3}} \tilde{f}_0 \quad \text{as} \quad \rho/(\nu^3/\tilde{\epsilon})^{\frac{1}{4}} \to 0, \tag{7.3}$$

thus providing a transition from the inertial $\rho^{\frac{4}{3}}$ to the dissipative ρ^2 as $\rho \to 0$. Let new variables

$$\rho = \rho/\rho_0, \quad \tau = t/\rho_0^{\frac{2}{3}} \tilde{\epsilon}^{-\frac{1}{3}}$$

be defined, where ρ_0 is the characteristic length scale of the initial conditions and $\rho_0^{\frac{2}{3}} \tilde{\epsilon}^{-\frac{1}{3}}$ is a turnaround time for inertial-range eddies. Then

$$\frac{\partial R_c}{\partial \tau} = \frac{1}{\zeta^2} \frac{\partial}{\partial \zeta} 2\zeta^2 [\zeta^{\frac{4}{3}} \tilde{f}(\zeta/\tau^{\frac{3}{2}}) + S\alpha\, F(\zeta/(\alpha S)^{\frac{3}{4}}, \tau/(\alpha S)^{\frac{1}{2}})] \frac{\partial R_c}{\partial \zeta} + 2\alpha \frac{1}{\zeta^2} \frac{\partial}{\partial \zeta} \zeta^2 \frac{\partial R_c}{\partial \zeta}, \tag{7.4}$$

where

$$\alpha = \mathscr{D}/\tilde{\epsilon}^{\frac{1}{3}} \rho_0^{\frac{4}{3}}.$$

It is assumed that $\rho_0 \gg (\nu^3/\tilde{\epsilon})^{\frac{1}{4}}$ and S is of order 1; therefore α is a small parameter. Equation (7.4) is to be solved with initial condition $R_c(\zeta, 0) \equiv R_{c0}(\zeta)$. The solution must have the property

$$\int_0^\infty \zeta^2 R_c(\zeta, t)\, d\zeta = \int_0^\infty \zeta^2 R_{c0}(\zeta)\, d\zeta. \tag{7.5}$$

An outer solution is sought in the form of a power series in the small parameter α; $R_c = R_c^{(0)} + \alpha R_c^{(1)} + \ldots$. The first term must therefore satisfy

$$\left.\begin{aligned} \frac{\partial R_c^{(0)}}{\partial \tau} &= \frac{1}{\zeta^2}\frac{\partial}{\partial \zeta} 2\zeta^{\frac{10}{3}} \tilde{f}(\zeta/\tau^{\frac{3}{2}}) \frac{\partial R_c^{(0)}}{\partial \zeta}, \\ R_c^{(0)} &= R_{c0} \quad \text{at} \quad \tau = 0. \end{aligned}\right\} \tag{7.6}$$

The general solution of this is not known. However since equations (7.6) and (5.20) are the same, the similarity solution, (5.24), may be adapted to this case. The solution is

$$R_c^{(0)}(\zeta, \tau) = A\tau^{-\frac{9}{2}} \exp\left\{-\frac{3}{4}\int_0^{\zeta/\tau^{\frac{3}{2}}} \frac{dy}{y^{\frac{1}{3}} \tilde{f}(y)}\right\}, \tag{7.7}$$

where

$$A = N \int_0^\infty \zeta^2 R_{c0}(\zeta)\, d\zeta.$$

This solution does not satisfy the exact initial conditions, but does satisfy (7.5). It is assumed that the exact solution will tend to equation (7.7) for large τ.

The similarity solution has the property

$$R_c^{(0)} \to A\tau^{-\frac{9}{2}}\left(1 - \frac{9}{8\tilde{f}_0}\left(\frac{\zeta}{\tau^{\frac{3}{2}}}\right)^{\frac{2}{3}} + \ldots\right) \tag{7.8}$$

as $\zeta \to 0$, which is clearly singular at $\zeta = 0$. This corner must be smoothed out by matching this outer solution to an inner solution which is valid near $\zeta = 0$.

An inner solution is obtained by first introducing an inner variable ξ defined by $\zeta = \xi\alpha^{\frac{3}{4}}$. The equation then takes the form

$$\alpha^{\frac{1}{2}}\frac{\partial R_c}{\partial \tau} = \frac{1}{\xi^2}\frac{\partial}{\partial \xi} 2\xi^2 [\xi^{\frac{4}{3}} \tilde{f}(\xi\alpha^{\frac{3}{4}}/\tau^{\frac{3}{2}}) + SF(\xi/S^{\frac{3}{4}}, \tau/(\alpha S)^{\frac{1}{2}})] \frac{\partial R_c}{\partial \xi} + \frac{2}{\xi^2}\frac{\partial}{\partial \xi}\xi^2\frac{\partial R_c}{\partial \xi}, \tag{7.9}$$

in which molecular and turbulent diffusion terms are all of the same order in α. The new variable scales lengths with $(\mathscr{D}^3/\tilde{\epsilon})^{\frac{1}{4}}$ instead of ρ_0.

When expressed in the new variables, equation (7.8) takes the form (inner expansion of outer expansion)

$$R_c^{(0)} = A\tau^{-\frac{9}{2}}\left(1 - \frac{9}{8\tilde{f}_0}\frac{\xi^{\frac{2}{3}}}{\tau}\alpha^{\frac{1}{2}} + \ldots\right), \tag{7.10}$$

which suggests that the inner solution should be a power series in $\alpha^{\frac{1}{2}}$. With

$$R_c = \phi^{(0)} + \alpha^{\frac{1}{2}}\phi^{(1)} + \ldots$$

one finds

$$\phi^{(0)} = \phi^{(0)}(\tau), \tag{7.11}$$

$$\frac{d\phi^{(0)}}{d\tau} = \frac{1}{\xi^2}\frac{\partial}{\partial \xi} 2\xi^2 [\xi^{\frac{4}{3}}\tilde{f}_0 + SF(\xi/S^{\frac{3}{4}}, \infty) + 1]\frac{\partial \phi^{(1)}}{\partial \xi}, \tag{7.12}$$

the latter having the solution

$$\phi^{(1)} = \frac{1}{6}\frac{d\phi^{(0)}}{d\tau}\int_0^\xi \frac{\xi' d\xi'}{1 + \xi'^{\frac{4}{3}}\tilde{f}_0 + SF(\xi'/S^{\frac{3}{4}}, \infty)} + C^{(1)}, \tag{7.13}$$

which corresponds to equation (6.18). One term matching of this inner expansion to the outer expansion (7.10), gives

$$\phi^{(0)}(\tau) = A\tau^{-\frac{9}{2}}. \tag{7.14}$$

One would need to carry one more term in the outer expansion to determine the constant $C^{(1)}$.

The main result is

$$\langle c^2 \rangle = N\rho_0^3 \int_0^\infty \zeta^2 R_{c0}(\zeta)\, d\zeta\, (\tilde{\epsilon}^{\frac{1}{3}}t)^{-\frac{9}{2}} + O(\alpha^{\frac{1}{2}}). \tag{7.15}$$

This has been obtained with the assumption $S = \nu/\mathscr{D} = O(1)$. Separate expansions have been made with $S \gg 1$ and $S \ll 1$ with the same result. The $S \ll 1$ expansion requires $\rho_0 \gg (\mathscr{D}^3/\tilde{\epsilon})^{\frac{1}{4}}$. The result for $S \ll 1$ is subject to some question since this parameter range is where the basic equation, (6.15), is not quite right. However, since the scaling is correct in this range it is likely that the result is valid here also.

It can be seen from (7.10) that the effective length scale of the scalar correlation increases like $t^{\frac{3}{2}}$ and will eventually be comparable to the size of the energy-containing eddies, at which time the present decay results will fail to be valid.

The experiments of Warhaft & Lumley are not perfectly suited to verify this decay rate because the heated wire spacing is comparable to the grid spacing and the turbulence is not stationary. However, the case with the smallest heated wire spacing, in which the wire spacing is equal to the grid spacing and the wires are located farthest downstream (where the scale of the turbulence has increased some), has the fastest decay rate, giving a decay like $t^{-3\cdot 2}$. In this case the energy spectrum has an approximate $k^{-\frac{5}{3}}$ range of about one decade and the measured scalar spectrum peaks in this range. It is felt that part of the discrepancy between $t^{-3\cdot 2}$ and $t^{-4\cdot 5}$ can be accounted for by the non-stationarity of the flow. If one replaces $\epsilon^{\frac{1}{3}}t$ by $\int_0^t \epsilon^{\frac{1}{3}}\,dt$ (an *ad hoc* procedure) and uses the experimental $\epsilon(t)$ one obtains decay exponents between 3·1 and 2·5 at the appropriate distance downstream of the grid.

A more complete treatment of turbulent diffusion in decaying isotropic homogeneous turbulence is within the realm of the general equations described in this paper and will be considered in later work.

It is of some interest to calculate the scalar spectrum which corresponds to the outer solution $R_c^{(0)}$. This has been done by using equation (7.7), with the approximation for $\tilde{f}$ given by (4.47), in equation (6.14). The resulting expression is

$$\left.\begin{aligned} G^{(0)}(k,t) &= \frac{2}{\pi}\rho_0^3\, A(\tilde{\epsilon}^{\frac{1}{3}}t)^{-3}\,\tilde{G}(x), \\ \tilde{G}(x) &= \int_0^\infty xY \sin xY \exp\{-3\cdot 52(Y^{\frac{2}{3}} + 0\cdot 165\, Y^{\frac{4}{3}})\}\, dY, \\ x &= k(\tilde{\epsilon}^{\frac{1}{3}}t)^{\frac{3}{2}}. \end{aligned}\right\} \tag{7.16}$$

The function $\tilde{G}$ has been computed and is presented in figure 5. It behaves like x^2 for small x and like $x^{-\frac{5}{3}}$ for large x. Corrections corresponding to the inner expansion would be at large x, beyond the $x^{-\frac{5}{3}}$ range. Since the wavenumber is scaled with $t^{\frac{3}{2}}$ this shows that the peak in the spectrum moves toward small wavenumber like $t^{-\frac{3}{2}}$ and the width of the peak also becomes smaller like $t^{-\frac{3}{2}}$. The magnitude of the peak decreases like t^{-3}. This spectrum is very similar in shape to those presented by Warhaft & Lumley.

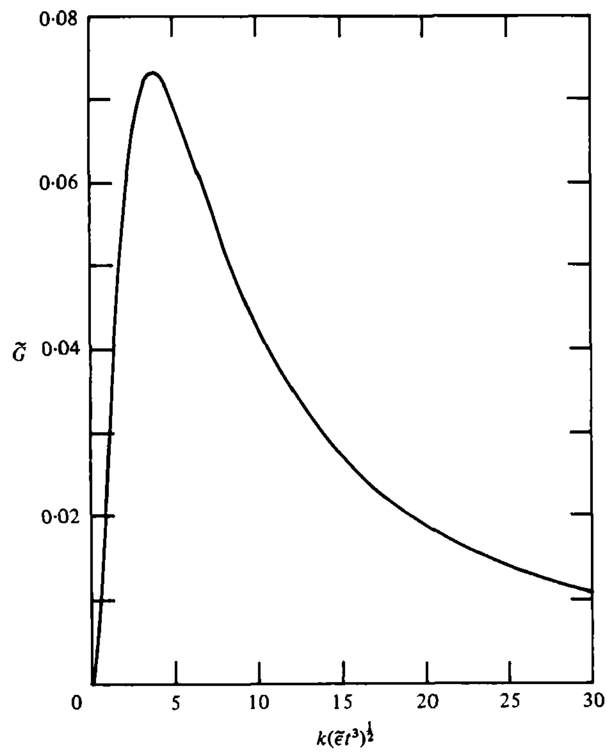

FIGURE 5. The scalar spectrum after inertial range injection, calculated from the outer similarity solution.

8. Conclusion

A self-consistent theory of pair dispersion and scalar diffusion in isotropic, homogeneous turbulence has been presented in which the only adjustable parameters are those required to specify the energy spectrum. The general equations have been applied to two- and three-dimensional stationary turbulence in order to compare the results with experiments. The most extensive experimental results for which comparisons could be made were for the scalar spectrum in the viscous–convective and inertial–convective wavenumber ranges where the quantitative agreement was satisfactory. This directly supports the results for the turbulent pair diffusion coefficient. Other comparisons of a more qualitative nature were also made. These included the Richardson 'three-halves power' law for the separation of particles in inertial range three-dimensional turbulence and the corresponding exponential separation in inertial range two-dimensional turbulence.

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